

Consumption-Portfolio Choice with Asset Tastes

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Abstract: This paper studies a continuous-time consumption–portfolio choice problem in which an investor has preferences for certain risky assets (“asset tastes”). A prominent recent example is the preference for green over brown assets. In a framework with a risk-free bond and two risky assets, one favored and one non-favored, we derive solutions for finite and infinite horizons. The risky assets are exposed to both diffusive and jump risk. The model can be interpreted as a dynamic resource-allocation problem under uncertainty in which alternatives differ in their financial and nonfinancial attributes. We establish existence and uniqueness of the solutions and provide verification theorems for optimality. Our model generates two key effects that shape optimal portfolio demand: Preferences for a particular asset increase exposure to that asset, which we call the preference effect. At the same time, demand for the non-favored asset is crowded out, generating a substitution effect. While the directions of these effects are intuitive, their magnitudes depend critically on market characteristics. In particular, the sensitivity (“beta”) of the favored asset to the non-favored asset determines the strength of the crowding-out effect. The total risky share increases when this sensitivity is below one and decreases when it exceeds one. Portfolio outcomes are thus jointly determined by preference-based features, such as the strength of asset tastes, and market-based features, such as the comovement between favored and non-favored assets. A numerical calibration based on U.S. data for green and brown stocks illustrates the theoretical results and highlights their quantitative implications.

Keywords: consumption-portfolio choice, resource allocation, asset tastes, green vs. brown assets, stochastic control

JEL subject codes: G11, C61

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1 Introduction

Several studies document that investors hold stocks not only for financial returns but also for non-pecuniary motives, such as home bias toward domestic stocks (French and Poterba, 1991; Tesar and Werner, 1995), environmental or social values such as preferences for “green” assets (Barber et al., 2021; Baker et al., 2022; Giglio et al., 2025), or loyalty to a company (Cohen, 2009). Such investors exhibit preferences or “tastes” for certain assets or asset types, leading them to overweight these assets relative to the predictions of classical portfolio theory (Baker et al., 2022; Giglio et al., 2025) and potentially distort equilibrium asset prices (Fama and French, 2007; Pedersen et al., 2021).

While many authors have incorporated preferences for particular assets into theoretical models in various contexts, most rely on stylized frameworks with static optimization problems (see, e.g., Andreoni, 1990; Fama and French, 2007; Pástor et al., 2021; Pedersen et al., 2021; Landier and Lovo, 2025, among others). Formal analysis of their implications for dynamic portfolio choice remains sparse.

Following Fama and French (2007), who introduced the notion of “tastes” for particular assets and study their general-equilibrium implications, we incorporate such preferences (“asset tastes”) into a continuous-time consumption-portfolio choice model with multiple stocks and jump risk. We solve for the optimal consumption and portfolio decisions, establish optimality through verification theorems, and study how asset tastes affect overall stock exposure and the composition of the risky portfolio.

Including multiple risky assets and jump risk is essential for understanding how preference biases shape portfolio choice in realistic financial markets. Consider, for example, an agent who favors green over brown stocks. Such a setting requires at least two risky assets, one favored and one non-favored, which may differ substantially in their risk exposures. Brown firms may be more exposed to abrupt policy, regulatory, technological, or demand shocks that generate discontinuous changes in fundamentals and stock prices, while green stocks are often modeled as favored assets. How preference-driven demand interacts with jump exposures across risky assets is thus an important research question.

From a technical point of view, solving our model is demanding because some of the equations characterizing the solution become implicit in the presence of jump risk. Consequently, several results are harder to prove than in models with diffusive risk only (e.g., Kraft and Weiss, 2019). For instance, in the infinite-horizon case, the optimal demand for the favored stock is characterized only implicitly by the zeros of a nonlinear function rather than explicitly by the roots of a polynomial. This significantly complicates some of the proofs.

We emphasize that preferences for particular stocks have richer economic consequences than preferences for entire asset classes, such as cash. With a preference for cash, the investor increases her cash holdings and reduces her demand for all risky stocks. The reverse holds if she has a preference for stocks in general. In our framework, however, preferences affect only some stocks and thus distort portfolio composition within an asset class. As expected, the agent increases her demand for

the favored asset (preference effect) and reduces her demand for the non-favored asset (substitution effect).

Additionally, unlike in models with preferences for entire asset classes, we identify a new effect on overall risky exposure that can go in *both* directions. Although the demand for the non-favored asset is always crowded out, the strength of this crowding-out effect determines whether overall risky exposure increases or decreases. We measure this exposure by the total risky share, defined as the sum of all risky portfolio shares.

We show that the strength of the crowding-out effect is determined by the sensitivity (‘beta’) of the favored asset to the non-favored asset. If this sensitivity is below one, overall stock exposure increases. Otherwise, it decreases, which may at first seem surprising. The latter case can arise when the assets have heterogeneous risk profiles and the favored asset is ‘riskier’ in this beta sense. Intuitively, the investor is aware of her bias and offsets its effect on her overall risky position by reducing her exposure to the non-favored asset. The riskier the favored asset, the larger this reduction must be. Since beta is a normalized measure of the shared risk between the favored and non-favored assets, the investor adjusts her portfolio according to this shared risk component.

Our paper relates to the literature on dynamic resource allocation and multi-attribute portfolio choice. Portfolio optimization can be interpreted as the allocation of scarce resources across uncertain alternatives as described by Salo et al. (2024). Dynamic expected-utility portfolio problems have been studied directly in Haugh et al. (2006). A related literature extends the classical risk–return framework by allowing investment alternatives to differ along additional attributes. In particular, Hallerbach et al. (2004), Ballesterio et al. (2012), and Hirschberger et al. (2013) incorporate social responsibility or sustainability as an additional criterion in portfolio selection. Our model combines these two strands by introducing preferences for asset characteristics into a dynamic continuous-time allocation problem. In our setting, green and brown assets differ not only in their financial payoffs but also in an environmental attribute that enters the investor’s preferences. Related OR work considers climate-related portfolio allocation and the transition toward low-carbon investments, as in Benedetti et al. (2021); Semmler et al. (2024).

Our paper also builds on the classical continuous-time portfolio-choice framework of Merton (1969, 1971). Subsequent work derives analytic portfolio rules when both returns and volatility can jump Liu et al. (2003), shows how systemic co-jumps reduce diversification benefits Das and Uppal (2004), and applies robust control to study ambiguity and model uncertainty associated with jumps Anderson et al. (2003). Other extensions include hidden Markov dynamics for jump intensities Bäuerle and Rieder (2007), closed-form solutions based on an orthogonal decomposition Aït-Sahalia et al. (2009), and contagion through mutually exciting jumps Aït-Sahalia and Hurd (2016). Beyond the standard continuous-time dynamic-programming approach, Bäuerle and Rieder (2009) derive optimal portfolio weights in a pure-jump market using a discrete-time Markov decision process.

Finally, our paper is related to models in which investors derive utility directly from particular portfolio positions. Building on Kraft and Weiss (2019), who study a consumption–portfolio problem with preferences for cash, Tian (2026) considers a continuous-time consumption–portfolio problem in which the investor has preferences for total exposure to risky stocks. That model contains a single risky asset and no jump risk. By contrast, our framework contains multiple risky assets with jump risk and allows preferences to distinguish between favored and non-favored assets within the same asset class.

The remainder of the paper is structured as follows. Section 2 introduces the baseline model, solves the problem by dynamic programming, provides verification theorems, and discusses the economic intuition. Section 3 examines the portfolio-choice implications. Section 4 calibrates the model to U.S. data and presents numerical examples in a “green versus brown” investing context. Section 5 concludes. The Appendix contains the proofs and additional material.

2 Model

2.1 Setup

We consider a canonical consumption-portfolio choice problem as in Merton (1969, 1971). The agent can invest in a bond and two stocks, which may represent, for example, two non-overlapping market segments or a stock index and one of its subgroups. For simplicity, we refer to them as the first and second stock. Their price dynamics are given by¹

$$\begin{aligned} dM &= Mrdt, \\ dS &= S[\mu dt + \sigma dW] - S_-[L_1 dN + \widehat{L}_1 d\widehat{N}], \\ d\widetilde{S} &= \widetilde{S}[\widetilde{\mu} dt + \widetilde{\sigma}(\rho dW + \sqrt{1 - \rho^2} d\widehat{W})] - \widetilde{S}_-[L_2 d\widetilde{N} + \widehat{L}_2 d\widehat{N}], \end{aligned}$$

where r is the interest rate, while μ , $\widetilde{\mu}$ and $\sigma > 0$, $\widetilde{\sigma} > 0$ denote the constant diffusive drifts and volatilities of the two stocks. W and $\widehat{W}$ are independent Brownian motions, and ρ is the constant instantaneous correlation between the stocks’ Brownian components. Further, N , $\widetilde{N}$ and $\widehat{N}$ are independent Poisson processes with constant intensities $\lambda > 0$, $\widetilde{\lambda} > 0$, and $\widehat{\lambda} > 0$. The processes N and $\widetilde{N}$ capture idiosyncratic jumps, while $\widehat{N}$ captures systematic jump risk affecting both stocks. The corresponding jump sizes satisfy $0 < L_1 < 1$, $0 < L_2 < 1$, $0 < \widehat{L}_1 < 1$, and $0 < \widehat{L}_2 < 1$.² The agent’s wealth X thus satisfies the dynamic budget constraint

$$\begin{aligned} dX &= X[(r + \eta\pi + \widetilde{\eta}\widetilde{\pi})dt + (\sigma\pi + \rho\widetilde{\sigma}\widetilde{\pi})dW + \sqrt{1 - \rho^2}\widetilde{\sigma}\widetilde{\pi}d\widehat{W}] \\ &\quad - X_-[L_1\pi_-dN + L_2\widetilde{\pi}_-d\widetilde{N} + (\widehat{L}_1\pi_- + \widehat{L}_2\widetilde{\pi}_-)d\widehat{N}] - cdt. \end{aligned} \tag{2.1}$$

¹For notational convenience, we omit time indices whenever no ambiguity arises.

²To avoid tedious case distinctions, we assume that all jump sizes are strictly positive. The online appendix discusses the special case without jumps.

Here $\eta := \mu - r$ and $\tilde{\eta} := \tilde{\mu} - r$ are the stocks' excess return parameters. Further, π and $\tilde{\pi}$ denote the proportions of wealth invested in the two stocks, and c denotes consumption. The agent maximizes expected utility from consumption and terminal wealth while exhibiting preferences for the first stock:

$$\sup_{c, \pi, \tilde{\pi}} \mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t \pi_t) dt + \omega e^{-\delta T} U(X_T, X_T \pi_T) \right], \quad (2.2)$$

where $X_t \pi_t$ denotes the amount invested in the first stock at time t , $\delta > 0$ is the agent's time preference rate, and $\omega > 0$ measures the strength of her bequest motive. We further assume that

$$u(c, \ell) = \frac{1}{1-\gamma} (c^\alpha \ell^{1-\alpha})^{1-\gamma}, \quad U(x, \ell) = \frac{1}{1-\gamma} (x^{\tilde{\alpha}} \ell^{1-\tilde{\alpha}})^{1-\gamma},$$

where $\gamma > 0$ denotes the coefficient of relative risk aversion.³ The constants $\alpha \in (0, 1)$ and $\tilde{\alpha} \in (0, 1]$ are the weights the agent assigns to consumption and wealth relative to her investment in the first stock.⁴ Accordingly, we call the first stock the *favoured* stock and the second the *non-favoured* stock.⁵ Throughout this section, we impose the following

Assumption 2.1.

- (a) *The coefficient of relative risk aversion is greater than one, i.e., $\gamma > 1$.*
- (b) *The instantaneous correlation is non-negative, $\rho \in [0, 1)$.*
- (c) *Without preferences for stock, the first stock's optimal demand satisfies $m \in (0, \frac{1}{L_1})$.*⁶

Condition (a) is imposed for economic reasons.⁷ Conditions (b) and (c) simplify the proofs and are typically satisfied under empirically reasonable parameterizations, as shown in Section 4.

2.2 Hamilton-Jacobi-Bellman Equation

The Hamilton-Jacobi-Bellman equation (HJB) associated with the maximization problem (2.2) subject to the dynamic budget constraint (2.1) is

$$\begin{aligned} 0 = \sup_{c, \pi, \tilde{\pi}} \{ & G_t - \delta G + x(r + \pi\eta + \tilde{\pi}\tilde{\eta})G_x - cG_x + 0.5x^2G_{xx}[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ & + \lambda[G(t, x(1 - L_1\pi)) - G(t, x)] + \tilde{\lambda}[G(t, x(1 - L_2\tilde{\pi})) - G(t, x)] \\ & + \hat{\lambda}[G(t, x(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})) - G(t, x)] + u(c, x\pi) \}. \end{aligned} \quad (2.3)$$

³The *effective* coefficient of relative risk aversion is $RRA = \gamma - (\gamma - 1)(1 - \alpha) = 1 + \alpha(\gamma - 1)$. For brevity, we still refer to γ as the coefficient of relative risk aversion.

⁴For $\alpha = 1$ and $\tilde{\alpha} = 1$, the canonical model obtains. To avoid tedious case distinctions, we exclude $\alpha = 1$.

⁵We avoid the term “preferred stock”, which usually refers to a distinct class of equity securities.

⁶The solution to the problem without preferences for stock is in the Online Appendix F. In particular, m is constant.

⁷See, e.g., Munk (2013), pp. 175–177. Notice that the agent's effective risk aversion is even lower than γ .

where

$$G(t, x) := \sup_{c, \pi, \tilde{\pi}} \left\{ e^{\delta t} \mathbb{E}_t \left[\int_t^T e^{-\delta s} u(c_s, X_s \pi_s) ds + \omega e^{-\delta T} U(X_T, X_T \pi_T) \right] \right\}$$

denotes the agent's indirect utility, and the terminal condition is $G(T, x) = \omega U(x, x\pi(T))$.⁸ In equation (2.3), the jump terms are problematic because they involve G evaluated at different levels of x . It is thus useful to exploit the following conjecture:

$$G(t, x) = \frac{1}{1-\gamma} x^{1-\gamma} f(t)^k \quad (2.4)$$

where k is a constant to be chosen. The terminal condition for f is then

$$f(T) = \omega^{\frac{1}{k}} \pi(T)^{\frac{(1-\alpha)(1-\gamma)}{k}}. \quad (2.5)$$

Given this conjecture, it is obvious that $G(t, \nu x) = \nu^{1-\gamma} G(t, x)$ for any $\nu \in (0, \infty)$. Thus, the HJB equation can be rewritten as

$$\begin{aligned} 0 = \sup_{c, \pi, \tilde{\pi}} \{ & G_t - \delta G + x(r + \pi\eta + \tilde{\pi}\tilde{\eta})G_x - cG_x + 0.5x^2 G_{xx}[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1-\rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ & + \overline{D}(\pi, \tilde{\pi})G + u(c, x\pi) \}, \end{aligned}$$

where we define $\overline{D}(\pi, \tilde{\pi}) := \lambda[(1 - L_1\pi)^{1-\gamma} - 1] + \tilde{\lambda}[(1 - L_2\tilde{\pi})^{1-\gamma} - 1] + \hat{\lambda}[(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{1-\gamma} - 1]$. Intuitively, $\overline{D}(\pi, \tilde{\pi})$ represents the expected loss in the agent's total utility due to jumps when fractions π and $\tilde{\pi}$ of wealth are invested in the two stocks. Its partial derivatives with respect to π and $\tilde{\pi}$ are

$$\begin{cases} \frac{\partial \overline{D}(\pi, \tilde{\pi})}{\partial \pi} &= (\gamma - 1)[\lambda L_1(1 - L_1\pi)^{-\gamma} + \hat{\lambda} \hat{L}_1(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma}] =: (\gamma - 1)\overline{A}(\pi, \tilde{\pi}), \\ \frac{\partial \overline{D}(\pi, \tilde{\pi})}{\partial \tilde{\pi}} &= (\gamma - 1)[\tilde{\lambda} L_2(1 - L_2\tilde{\pi})^{-\gamma} + \hat{\lambda} \hat{L}_2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma}] =: (\gamma - 1)\overline{B}(\pi, \tilde{\pi}). \end{cases}$$

These partial derivatives capture the marginal utility loss from increasing π or $\tilde{\pi}$ due to greater exposure to jump risk. The first-order condition (FOC) with respect to c is

$$c = \alpha^{\frac{1}{\kappa}} (x\pi)^{\frac{(\alpha-1)(\gamma-1)}{\kappa}} G_x^{-\frac{1}{\kappa}}, \quad (2.6)$$

where $\kappa := 1 + \alpha(\gamma - 1)$ is the agent's *effective* relative risk aversion. By Assumption 2.1 (a), we have $\kappa > 0$. Moreover, the first-order condition with respect to π is

$$\eta G_x + \sigma x G_{xx}(\sigma\pi + \rho\tilde{\sigma}\tilde{\pi}) + (\gamma - 1) \frac{\overline{A}G}{x} + (1 - \alpha)[c^\alpha (x\pi)^{1-\alpha}]^{-\gamma} c^\alpha (x\pi)^{-\alpha} = 0. \quad (2.7)$$

⁸We assume that the optimal demand for the first stock only depends on time. Later we verify this assumption (see (2.12) and (2.15)).

Finally, the first-order condition with respect to $\tilde{\pi}$ is given by

$$\tilde{\pi} = -\frac{\tilde{\eta}G_x}{\tilde{\sigma}^2xG_{xx}} - \rho\frac{\sigma}{\tilde{\sigma}}\pi + \frac{(1-\gamma)\bar{B}G}{\tilde{\sigma}^2x^2G_{xx}}. \quad (2.8)$$

(2.6) and (2.8) explicitly express c and $\tilde{\pi}$ in terms of π (except for the dependency of $\bar{A}$ and $\bar{B}$ on $\tilde{\pi}$). Substituting these equations back to the HJB equation, we can eliminate c and $\tilde{\pi}$:

$$\begin{aligned} 0 = \sup_{\pi, \tilde{\pi}} \{ & G_t - \delta G + rxG_x - \frac{1}{2} \frac{\tilde{\eta}^2 G_x^2}{\tilde{\sigma}^2 G_{xx}} + \frac{1}{2} \frac{(1-\gamma)^2 \bar{B}^2 G^2}{\tilde{\sigma}^2 x^2 G_{xx}} + \bar{D}G + (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})x\pi G_x \\ & + \frac{1}{2}(1-\rho^2)\sigma^2 x^2 \pi^2 G_{xx} + K(x\pi)^{\frac{(\alpha-1)(\gamma-1)}{\kappa}} G_x^{\frac{\kappa-1}{\kappa}} \}, \end{aligned} \quad (2.9)$$

where $K := \frac{1}{1-\gamma}\alpha^{\frac{\alpha(1-\gamma)}{\kappa}} - \alpha^{\frac{1}{\kappa}} = -(\frac{1}{\gamma-1} + \alpha)\alpha^{\frac{1}{\kappa}-1} < 0$ by Assumption 2.1 (a). The corresponding first-order condition with respect to π can be simplified by substituting (2.6) and (2.8) into (2.7):

$$(\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta}) + (1-\rho^2)\sigma^2\pi\frac{xG_{xx}}{G_x} + \frac{1-\gamma}{\tilde{\sigma}}(\rho\sigma\bar{B} - \tilde{\sigma}\bar{A})\frac{G}{xG_x} + \frac{1-\alpha}{\alpha}\alpha^{\frac{1}{\kappa}}(x^\gamma\pi^\gamma G_x)^{-\frac{1}{\kappa}} = 0. \quad (2.10)$$

Applying the conjecture (2.4), equation (2.6) becomes

$$\frac{c}{x} = \alpha^{\frac{1}{\kappa}}(\pi^{\gamma-\kappa}f^k)^{-\frac{1}{\kappa}}, \quad (2.11)$$

and equation (2.10) becomes

$$(1-\rho^2)\gamma\sigma^2\pi - [(\eta - \bar{A}) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - \bar{B})] = \frac{1-\alpha}{\alpha}\alpha^{\frac{1}{\kappa}}(\pi^\gamma f^k)^{-\frac{1}{\kappa}}, \quad (2.12)$$

and finally, equation (2.8) becomes⁹

$$\tilde{\pi} = \frac{\tilde{\eta} - \bar{B}}{\gamma\tilde{\sigma}^2} - \rho\frac{\sigma}{\tilde{\sigma}}\pi. \quad (2.13)$$

To solve the model, we initially assume that the following statements concerning our conjecture (2.4) hold. Section 2.5 provides the corresponding verification theorems.

Assumption 2.2.

- (a) The value function has a representation of the form (2.4) with $f \in C^1$, $f > 0$ and $k \in \mathbb{R}$;
- (b) The FOCs (2.11), (2.12), and (2.13) determine the optimal consumption and stock demands;
- (c) If the horizon is infinite, an appropriate transversality condition holds.

Given Assumption 2.2, we can show global concavity of our problem.

⁹Note that this coincides with the first-order condition for the second stock in the problem without preferences for stock (see (F.5) in Appendix F for detail) because (for a given π) the optimal investment strategy for the second stock is not distorted by the special preference considered in our model.

Proposition 2.3. (*Global concavity*). Under conjecture (2.4) and Assumption 2.2 (i), the function

$$H(c, \pi, \tilde{\pi}) := x(\pi\eta + \tilde{\pi}\tilde{\eta})G_x - cG_x + 0.5x^2G_{xx}[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ + \overline{D}(\pi, \tilde{\pi})G + u(c, x\pi)$$

is strictly concave for all $(c, \pi, \tilde{\pi}) \in (0, \infty) \times (0, \frac{1}{L_1}) \times (-\infty, \frac{1}{L_2})$ and satisfies $\hat{L}_1\pi + \hat{L}_2\tilde{\pi} < 1$, i.e., the solutions to the FOCs (2.11), (2.12) and (2.13) are a global maximum of the HJB equation (2.3).

Based on the first-order condition (2.13) for $\tilde{\pi}$, the following proposition establishes the existence, uniqueness, and differentiability of the optimal demand for the second stock:

Proposition 2.4. (*Existence, uniqueness and differentiability of the second stock's optimal demand*). Under Assumption 2.1, for every $\pi \in \mathbb{R}$ fixed, equation (2.13) always has a unique solution for $\tilde{\pi}$ on $(-\infty, \min(\frac{1}{L_2}, \frac{1-\hat{L}_1\pi}{\hat{L}_2}))$. Moreover, a unique and continuously differentiable implicit function $\tilde{\pi}(\pi)$ exists for every $\pi \in \mathbb{R}$ and satisfies

$$\frac{d\tilde{\pi}}{d\pi} = -\frac{\rho\sigma\tilde{\sigma} + \hat{\lambda}\hat{L}_1\hat{L}_2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}}{\tilde{\sigma}^2 + \hat{\lambda}L_2^2(1 - L_2\tilde{\pi})^{-\gamma-1} + \hat{\lambda}\hat{L}_2^2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}} < 0. \quad (2.14)$$

Proposition 2.4 allows us to express $\tilde{\pi}$ as a function of π , thereby reducing the problem to a system of equations in π only. Accordingly, we redefine the functions $\overline{D}$, $\overline{A}$, and $\overline{B}$ by substituting the implicit function $\tilde{\pi}(\pi)$ defined in Proposition 2.4:

$$A(\pi) := \overline{A}(\pi, \tilde{\pi}(\pi)), \quad B(\pi) := \overline{B}(\pi, \tilde{\pi}(\pi)), \quad D(\pi) := \overline{D}(\pi, \tilde{\pi}(\pi)).$$

In the following, we use these definitions so that the relevant equations depend on π only. Further, equation (2.12) yields more precise information about the magnitude of the optimal demand for the first stock, as summarized in the following proposition:

Proposition 2.5. (*Size of the optimal demand for the first stock*). Under Assumptions 2.1 and 2.2, the agent's optimal demand for the first stock satisfies the following inequalities: For all $t \in [0, T]$,

$$(a) \pi^*(t) > \frac{1}{(1-\rho^2)\gamma\sigma^2}[(\eta - A(\pi^*(t))) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\pi^*(t)))]; \quad (b) \pi^*(t) > m; \quad (c) \pi^*(t) < \frac{1}{L_1},$$

where m is the optimal demand for the first stock in the case without preferences for stock.

The upper bound in (c) ensures non-negative wealth, as in Liu et al. (2003). The lower bound in (a) follows from the agent's direct preference for the first stock. Relative to the standard case without stock-specific preferences, the agent thus invests a larger fraction of her wealth in the first stock.

By substituting (2.4) into the HJB equation (2.9), we obtain the reduced HJB equation as

$$0 = \sup_{\pi} \left\{ \frac{k}{1-\gamma} f_t + \left[r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + (\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta})\pi - \frac{1}{2}(1-\rho^2)\gamma\sigma^2\pi^2 - \frac{D}{\gamma-1} - \frac{B^2}{2\gamma\tilde{\sigma}^2} \right] f + K\pi^{\frac{\kappa-\gamma}{\kappa}} f^{1-\frac{\kappa}{\kappa}} \right\}. \quad (2.15)$$

The problem has now been reduced to the HJB equation (2.15) and its first-order condition (2.12). These two equations form a coupled system that is independent of wealth x . Hence, the separation is valid, and f and π are indeed functions of time only. Substituting (2.12) into (2.15) yields¹⁰

$$\begin{aligned} \frac{k}{1-\gamma} f_t + \left\{ r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + (\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta})\pi - \frac{1}{2}(1-\rho^2)\gamma\sigma^2\pi^2 - \frac{D}{\gamma-1} - \frac{B^2}{2\gamma\tilde{\sigma}^2} \right. \\ \left. + \tilde{K}[(1-\rho^2)\gamma\sigma^2\pi - [(\eta-A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)]\pi] \right\} f = 0, \end{aligned} \quad (2.16)$$

where $\tilde{K} := K \frac{\alpha}{1-\alpha} \alpha^{-\frac{1}{\kappa}} = \frac{\kappa}{(1-\gamma)(1-\alpha)} < 0$ by Assumption 2.1 (a). Choosing $k = \kappa$, the first-order condition for π (2.12) becomes

$$f = \frac{\frac{1-\alpha}{\alpha} \alpha^{\frac{1}{\kappa}}}{[(1-\rho^2)\gamma\sigma^2\pi - [(\eta-A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)]\pi^{\frac{\gamma}{\kappa}}}. \quad (2.17)$$

Equation (2.17) establishes a one-to-one relation between f and π , so that the optimal demand for the first stock determines the value function. Substituting this relation into equation (2.11) and eliminating f , the optimal consumption-wealth ratio becomes

$$\frac{c}{x} = \frac{\alpha}{1-\alpha} \left[(1-\rho^2)\gamma\sigma^2\pi - [(\eta-A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)] \right] \pi. \quad (2.18)$$

Given Proposition 2.5, equation (2.18) shows that the optimal consumption-wealth ratio is always positive and increases monotonically with π .¹¹ This ratio is also proportional to the ratio of the preference weights α and $1-\alpha$.

Equations (2.13), (2.17), and (2.18) express all relevant quantities in terms of π . It remains to solve for π , which is characterized by the ordinary differential equation (ODE) (2.16) together with the terminal condition (2.5). Combining (2.17) and (2.16) yields the following result:

Proposition 2.6. *(ODE characterization of the optimal demand for the first stock). The optimal demand for the first stock is given by the solution to the following ODE:*

$$\frac{d\pi}{dt} = -\frac{\gamma-1}{\kappa} \frac{P(\pi)\pi\left[\pi - \frac{(\eta-A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}\right]}{\pi + \frac{\gamma}{\kappa}\left[\pi - \frac{(\eta-A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}\right] + \frac{\pi}{(1-\rho^2)\gamma\sigma^2}\left[\frac{dA}{d\pi} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi}\right]}, \quad (2.19)$$

¹⁰Unless otherwise stated, all equations are now evaluated at the optimal strategies $c = c^*$, $\pi = \pi^*$, and $\tilde{\pi} = \tilde{\pi}^*$.

¹¹Note that $A - \rho \frac{\sigma}{\tilde{\sigma}} B$ is monotonically increasing in π . See equation (A.3).

where $\frac{dA}{d\pi} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi}$ is given by equation (A.3), and $P(\pi)$ is a nonlinear function with respect to π :

$$P(\pi) = r + \frac{\delta}{\gamma - 1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + (\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta})\pi - \frac{1}{2}(1 - \rho^2)\gamma\sigma^2\pi^2 - \frac{D}{\gamma - 1} - \frac{B^2}{2\gamma\tilde{\sigma}^2} \\ + \tilde{K}(1 - \rho^2)\gamma\sigma^2[\pi - \frac{(\eta - A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B)}{(1 - \rho^2)\gamma\sigma^2}]\pi.$$

Given the above ODE characterization, we make the following assumption on the function P :

Assumption 2.7. Assume that $P(m) > 0$, or equivalently:

$$r > -\frac{\delta}{\gamma - 1} - \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} - (1 - \rho^2)\gamma\sigma^2 \left[\frac{\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta}}{(1 - \rho^2)\gamma\sigma^2} - m \right] m - \frac{1}{2}(1 - \rho^2)\gamma\sigma^2 m^2 + \frac{D(m)}{\gamma - 1} + \frac{B(m)^2}{2\gamma\tilde{\sigma}^2},$$

where m is the optimal demand for the first stock in the case without preferences for stock.

As we will see in Section 2.3, Assumption 2.7 guarantees the existence of the model solution and is also crucial for the Verification Theorem 2.20 to hold. However, Section 4 shows that this assumption is only a weak constraint that easily holds under reasonable parameterizations of the model.

Proposition 2.6 characterizes the time dependence of the optimal demand for the first stock through the dynamics of π . In particular, the sign of its time derivative is of central interest. Recall from Proposition 2.5 that $\pi > m > 0$ and $\pi > \frac{1}{(1 - \rho^2)\gamma\sigma^2} [(\eta - A) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B)]$ always hold. Hence, the time dependence of π is determined by the sign of $P(\pi)$. Specifically:

Proposition 2.8. (Slope of the optimal demand for the first stock). Under Assumptions 2.1 and 2.2, the time derivative of $\pi(t)$ has the opposite sign of $P(\pi(t))$:

$$P(\pi) > 0 \quad \Leftrightarrow \quad \frac{d\pi}{dt} < 0 \quad \text{and} \quad P(\pi) < 0 \quad \Leftrightarrow \quad \frac{d\pi}{dt} > 0.$$

The terminal condition of the ODE (2.19) is given by combining (2.17) with (2.5) and letting $t = T$:

$$f(T) = \omega^{\frac{1}{\kappa}} \pi(T)^{\frac{(1 - \tilde{\alpha})(1 - \gamma)}{\kappa}} = \frac{\frac{1 - \alpha}{\alpha} \alpha^{\frac{1}{\kappa}}}{[(1 - \rho^2)\gamma\sigma^2\pi(T) - [(\eta - A(\pi(T))) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\pi(T)))]]\pi(T)^{\frac{\gamma}{\kappa}}},$$

which can be simplified to

$$(1 - \rho^2)\gamma\sigma^2[\pi(T) - \frac{(\eta - A(\pi(T))) - \rho \frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\pi(T)))}{(1 - \rho^2)\gamma\sigma^2}] = \omega^{-\frac{1}{\kappa}} \frac{1 - \alpha}{\alpha} \alpha^{\frac{1}{\kappa}} \pi(T)^{-\frac{\kappa_{\tilde{\alpha}}}{\kappa}}, \quad (2.20)$$

where $\kappa_{\tilde{\alpha}} := 1 + \tilde{\alpha}(\gamma - 1) > 0$. By (2.13) and (2.18), the terminal optimal demand for the second stock and the terminal optimal consumption-wealth ratio are determined by $\pi(T)$. Hence, $\pi(T)$ pins down all terminal conditions. It remains to establish the existence, uniqueness, and range of $\pi(T)$:

Proposition 2.9. (*Existence, uniqueness and range of $\pi(T)$*). Under Assumptions 2.1 and 2.2, the terminal value of the optimal demand for the first stock $\pi(T)$ exists, is unique, and satisfies $\pi(T) > \frac{1}{(1-\rho^2)\gamma\sigma^2}[(\eta - A(\pi(T))) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\pi(T)))]$, $\pi(T) > m$, and $\pi(T) < \frac{1}{L_1}$.

Combining Propositions 2.6 and 2.9, we can now discuss the properties of the optimal strategies in the infinite- and finite-horizon cases.

2.3 Infinite Horizon

With an infinite horizon, time is no longer a state variable, and the optimal demands π and $\tilde{\pi}$, the optimal consumption-wealth ratio $\frac{c}{x}$, and f are all constant. Hence,

$$-\frac{\gamma-1}{\kappa} \frac{P(\pi)\pi[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}]}{\pi + \frac{\gamma}{\kappa}[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}] + \frac{\pi}{(1-\rho^2)\gamma\sigma^2}[\frac{dA}{d\pi} - \rho\frac{\sigma}{\tilde{\sigma}}\frac{dB}{d\pi}]} = 0,$$

or, equivalently, by Proposition 2.5,

$$\begin{aligned} P(\pi) = & r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})\pi - \frac{1}{2}(1-\rho^2)\gamma\sigma^2\pi^2 - \frac{D}{\gamma-1} - \frac{B^2}{2\gamma\tilde{\sigma}^2} \\ & + \tilde{K}(1-\rho^2)\gamma\sigma^2[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}]\pi = 0. \end{aligned}$$

We then rewrite $P(\pi)$ as follows:

$$P(\pi) = \hat{K}(\pi^2 + p(\pi)\pi + q) - \frac{D(\pi)}{\gamma-1} - \frac{B(\pi)^2}{2\gamma\tilde{\sigma}^2} + [A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi)]\pi \quad (2.21)$$

with¹² $\hat{K} := (\tilde{K} - \frac{1}{2})(1-\rho^2)\gamma\sigma^2$, $p(\pi) := \frac{1}{\tilde{K}}(1-\tilde{K})(\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta}) + \frac{\tilde{K}-1}{\tilde{K}}[A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi)]$, $q := \frac{1}{\tilde{K}}(r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2})$, where $\hat{K} < 0$ by Assumption 2.1 (a) and (b).

Importantly, the bounds on $\pi^*(t)$ established in Proposition 2.5(b) and (c) also apply in the infinite-horizon case, since their proof relies only on the first-order conditions for π and $\tilde{\pi}$ and is independent of time. Therefore, it suffices to restrict attention to the economically feasible domain $(m, \frac{1}{L_1})$ for the optimal demand for the first stock.

Further, due to the presence of jumps, the function P is no longer a polynomial as in Kraft and Weiss (2019), but a more general nonlinear function. Hence, its zeros generally do not admit closed-form solutions. Nevertheless, we can establish their existence and number. To this end, we first establish some basic properties of the function P :

¹²Note that we have explicitly indicated which terms are dependent on π .

Lemma 2.10. (*Properties of P*). The function P is well-defined, continuous, and sufficiently differentiable on $[m, \frac{1}{L_1})$, and satisfies

$$\lim_{\pi \nearrow \frac{1}{L_1}} P(\pi) = -\infty. \quad (2.22)$$

By Assumption 2.7, we have $P(m) > 0$. Combining this with (2.22) and the continuity of P on $[m, \frac{1}{L_1})$, the intermediate value theorem implies that P has at least one real zero in $(m, \frac{1}{L_1})$. However, uniqueness is not immediate. We thus impose the following sufficient condition, which guarantees a unique zero of P on the feasible domain $(m, \frac{1}{L_1})$ and is maintained throughout the subsequent analysis:

Assumption 2.11. (*Sufficient condition for the uniqueness of P 's zero*). Let m denote the optimal demand for the first stock without stock-specific preferences. Assume that

$$r \geq -\frac{\delta}{\gamma-1} - \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} - \left[A(m) - \rho \frac{\sigma}{\tilde{\sigma}} B(m) \right] m + \hat{K}m^2 + \frac{D(m)}{\gamma-1} + \frac{B(m)^2}{2\gamma\tilde{\sigma}^2}.$$

Assumptions 2.7 and 2.11 impose lower bounds on the interest rate or, equivalently, the agent's time preference rate. Section 4 shows that these conditions hold under reasonable parameterizations. We can now establish the uniqueness of the optimal demand for the first stock:

Proposition 2.12. (*Optimal demand for the first stock in infinite-horizon case*). Under Assumptions 2.1, 2.2, 2.7 and 2.11, the function P as defined in (2.21) satisfies:

- (a) For any $\pi_0 \in (m, \frac{1}{L_1})$ that satisfies $P(\pi_0) = 0$, it must hold that $P'(\pi_0) < 0$;
- (b) P has one unique real zero on $(m, \frac{1}{L_1})$, denoted by π^∞ , which is the unique optimal demand for the first stock in the infinite-horizon case.

Proposition 2.12 further implies the following property of the P function:

Corollary 2.13. (*Sign of P*). P is strictly positive on (m, π^∞) and strictly negative on $(\pi^\infty, \frac{1}{L_1})$.

2.4 Finite Horizon

With a finite horizon, we return to the ODE (2.19) and its terminal condition (2.20). If $\pi(T) = \pi^\infty$, then $\pi(t) = \pi^\infty$ for all $t \in [0, T]$. Hence, we focus on the case $\pi(T) \neq \pi^\infty$.

Moreover, we start by assuming $\pi(t) \in [\min(\pi(T), \pi^\infty), \max(\pi(T), \pi^\infty)]$, i.e., the optimal demand for the first stock lies between $\pi(T)$ and π^∞ . We thus restrict our attention on a relatively smaller domain of π , i.e., $\pi \in [m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2]$. Here, $\varepsilon_1 < \min(\pi(T) - m, \pi^\infty - m)$ and $\varepsilon_2 < \min(\frac{1}{L_1} - \pi(T), \frac{1}{L_1} - \pi^\infty)$ are two fixed positive real numbers that are sufficiently small to allow $\pi(T)$ given by (2.20) and π^∞ given by Proposition 2.12 to be two interior points in the restricted domain. Proposition 2.15 confirms that $\pi(t)$ indeed remains between $\pi(T)$ and π^∞ , so this restriction is without loss of generality.

Separating variables in the ODE (2.19) yields:

$$\frac{\pi + \frac{\gamma}{\kappa}[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\sigma}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}] + \frac{\pi}{(1-\rho^2)\gamma\sigma^2}[\frac{dA}{d\pi} - \rho\frac{\sigma}{\sigma}\frac{dB}{d\pi}]}{P(\pi)\pi[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\sigma}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}]}d\pi = \bar{K}dt, \quad (2.23)$$

where $\bar{K} := -\frac{\gamma-1}{\kappa} < 0$. Integrating both sides yields

$$\mathcal{G}(\pi) := \int \frac{\pi + \frac{\gamma}{\kappa}[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\sigma}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}] + \frac{\pi}{(1-\rho^2)\gamma\sigma^2}[\frac{dA}{d\pi} - \rho\frac{\sigma}{\sigma}\frac{dB}{d\pi}]}{P(\pi)\pi[\pi - \frac{(\eta-A)-\rho\frac{\sigma}{\sigma}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}]}d\pi = \bar{K}t + \check{K}, \quad (2.24)$$

where $\mathcal{G}$ denotes an antiderivative of the left-hand side of (2.23), and $\check{K}$ is a constant determined by the choice of $\mathcal{G}$.¹³

It remains to show that $\mathcal{G}$ is well-defined, continuously differentiable, and invertible on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$. Then, by (2.24), the finite-horizon optimal demand for the first stock is obtained by inverting $\mathcal{G}$ on the appropriate domain. By the inverse function theorem, $\mathcal{G}^{-1}$ is also well-defined and continuously differentiable, yielding a candidate solution to the ODE (2.19). The following proposition summarizes these results:¹⁴

Proposition 2.14. *(Well-definedness, differentiability and invertibility of $\mathcal{G}$). Under Assumptions 2.1, 2.2, 2.7 and 2.11, there exists a function $\mathcal{G}$ that satisfies:*

- (a) $\mathcal{G}$ is a well-defined real-valued function on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$;
- (b) $\mathcal{G} \in C^1$ on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$;
- (c) $\mathcal{G}(\pi) \rightarrow \infty$ for $\pi \nearrow \pi^\infty$ and $\pi \searrow \pi^\infty$;
- (d) $\mathcal{G}'(\pi) > 0$ for all $\pi \in [m + \varepsilon_1, \pi^\infty)$, and $\mathcal{G}'(\pi) < 0$ for all $\pi \in (\pi^\infty, \frac{1}{L_1} - \varepsilon_2]$;
- (e) $\mathcal{G}$ has two well-defined and continuously differentiable inverses $\mathcal{G}^{-1} \in C^1$ on $\pi \in [m + \varepsilon_1, \pi^\infty)$ and $\pi \in (\pi^\infty, \frac{1}{L_1} - \varepsilon_2]$, respectively.

Proposition 2.14 establishes that equation (2.24) is well-defined for $\pi \in [m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$. Hence, the optimal demand for the first stock can be obtained by inverting $\mathcal{G}$. Before doing so, however, some technical details remain to be addressed.

First, Proposition 2.14(e) yields two candidate inverse functions. However, $\pi(t)$ must be continuously differentiable by Assumption 2.2(a), Proposition 2.4, and (2.17). Since the two candidates have disjoint ranges, continuity rules out switching between them. Hence, the terminal condition $\pi(T)$ in (2.20) determines the relevant inverse: only the inverse whose range contains $\pi(T)$ can be the solution.

¹³The function $\mathcal{G}$ is defined only up to an additive constant. Without loss of generality, we set this constant to zero when defining $\mathcal{G}$ through two definite integrals in Proposition 2.14, since $\check{K}$ already serves the same purpose.

¹⁴See equation (B.1) for the detailed expression of $\mathcal{G}$.

Second, substituting $\pi = \pi(T)$ into (2.24) determines the constant $\check{K}$:

$$\mathcal{G}(\pi(T)) = \bar{K}T + \check{K} \quad \Leftrightarrow \quad \check{K} = -\bar{K}T + \mathcal{G}(\pi(T)).$$

Therefore, for all $t \in [0, T)$, we obtain

$$\mathcal{G}(\pi(t)) = -\bar{K}(T - t) + \mathcal{G}(\pi(T)). \quad (2.25)$$

Third, the value $-\bar{K}(T - t) + \mathcal{G}(\pi(T))$ indeed lies in the range of $\mathcal{G}$. Since $\bar{K} < 0$, we have $-\bar{K}(T - t) + \mathcal{G}(\pi(T)) > \mathcal{G}(\pi(T))$. Thus, by Proposition 2.14 (b) and (c) and the intermediate value theorem, there exists a π between $\pi(T)$ and π^∞ satisfying $\mathcal{G}(\pi) = -\bar{K}(T - t) + \mathcal{G}(\pi(T))$, which determines $\pi(t)$.

Finally, the preceding arguments allow us to invert $\mathcal{G}$ in (2.25) and obtain a well-defined, unique, and continuously differentiable optimal demand for the first stock:

Proposition 2.15. (*Uniqueness, size and slope of the optimal demand for the first stock in finite-horizon case*). Under Assumptions 2.1, 2.2, 2.7 and 2.11, we have:

(a) If $\pi(T) \neq \pi^\infty$, then $\pi(t)$ is uniquely defined by $\pi(t) = \mathcal{G}^{-1}(-\bar{K}(T - t) + \mathcal{G}(\pi(T)))$, where the function $\mathcal{G}$ is defined in equation (B.1);

(b) If $\pi(T) < \pi^\infty$, then $\pi(t) \in (\pi(T), \pi^\infty)$ and $\frac{d\pi}{dt} < 0$ for all $t \in [0, T)$;

(c) If $\pi(T) > \pi^\infty$, then $\pi(t) \in (\pi^\infty, \pi(T))$ and $\frac{d\pi}{dt} > 0$ for all $t \in [0, T)$;

(d) If $\pi(T) = \pi^\infty$, then $\pi(t) = \pi^\infty$ is a constant for all $t \in [0, T]$.

In any case, $\pi(t)$ is a smooth C^1 -function on $[0, T]$, which is also true for $f(t)$ by (2.17).

Proposition 2.15 shows that $\pi(T)$ and π^∞ determine the time dependence of the model and that the optimal demand for the first stock is always monotonic.

Finally, the following proposition shows that all optimal strategies in the finite-horizon case are monotonic in the strength of the bequest motive ω . In particular, a stronger bequest motive reduces the fraction of wealth consumed over time:

Proposition 2.16. (*Monotonicity of the optimal strategies with respect to bequest motive in finite-horizon case*). Suppose $\omega_1, \omega_2 \in (0, \infty)$ and satisfy $\omega_1 < \omega_2$. Then under Assumptions 2.1, 2.2, 2.7 and 2.11, the following inequalities hold for all $t \in [0, T]$:

(a) $\pi^{(\omega_1)}(t) \geq \pi^{(\omega_2)}(t)$;

(b) $\tilde{\pi}^{(\omega_1)}(t) \leq \tilde{\pi}^{(\omega_2)}(t)$;

(c) $(\frac{c}{x})^{(\omega_1)}(t) \geq (\frac{c}{x})^{(\omega_2)}(t)$,¹⁵

where the superscript (ω_i) denotes the optimal strategy associated with bequest motive ω_i , $i = 1, 2$.

¹⁵Here, we explicitly indicate the time dependency of the agent's optimal consumption-wealth ratio.

2.5 Verification Theorems

In this subsection, we verify Assumption 2.2. In particular, we show that the conjecture (2.4) is the value function,¹⁶ that $(c^*, \pi^*, \tilde{\pi}^*)$ is the optimal strategy, and that the required transversality condition holds in the infinite-horizon case. In the following, $X^{c, \pi, \tilde{\pi}}$ denotes the wealth process generated by a given, not necessarily optimal, strategy $(c, \pi, \tilde{\pi})$, while X^* denotes the wealth process generated by our candidate strategy $(c^*, \pi^*, \tilde{\pi}^*)$. First, we define the set of admissible strategies as follows:

Definition 2.17. (*Admissible strategies*). A strategy $(c, \pi, \tilde{\pi})$ is admissible if:

- (a) The processes c , π and $\tilde{\pi}$ are progressively measurable with respect to the filtration generated by the Brownian motions W and $\widehat{W}$ as well as the Poisson processes N , $\tilde{N}$ and $\widehat{N}$;
- (b) For all initial conditions $(t_0, x_0) \in [0, T] \times (0, \infty)$, the wealth equation (G.1) with $X_{t_0}^{c, \pi, \tilde{\pi}} = x_0$ has a path-wise unique solution $\{X_t^{c, \pi, \tilde{\pi}}\}_{t \in [t_0, T]}$;
- (c) $X^{c, \pi, \tilde{\pi}} \geq 0$;
- (d) $c > 0$ and $\pi > 0$;
- (e) $c \leq \bar{c}X^{c, \pi, \tilde{\pi}}$, $\pi \leq \bar{\pi} < \frac{1}{L_1}$, $\tilde{\pi} \leq \bar{\tilde{\pi}} < \frac{1}{L_2}$, and $1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi} \geq M > 0$ for constants $\bar{c}$, $\bar{\pi}$, $\bar{\tilde{\pi}}$, $\underline{\pi}$, $\underline{\tilde{\pi}}$ and M that can be different for different strategies.

The set of admissible strategies is denoted by $\mathcal{A}$.

Remark. (i) Conditions (a)–(c) hold if $\frac{c}{X^{c, \pi, \tilde{\pi}}}$, π , and $\tilde{\pi}$ are smooth deterministic functions;

(ii) If condition (d) is not satisfied, the utility functional is not well-defined;

(iii) Condition (e) ensures the integrability of c , π , and $\tilde{\pi}$. While alternative conditions are possible, this boundedness restriction is economically natural. In particular, it excludes strategies for which $1 - L_1\pi$, $1 - L_2\tilde{\pi}$, or $1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi}$ can become arbitrarily close to zero.

The following theorem summarizes our results for the candidate value function and optimal strategy derived in the previous subsections:

Theorem 2.18. (*Candidates for value function and optimal strategy*). Under Assumptions 2.1, 2.7 and 2.11, we have:

- (a) The candidate for the first stock's optimal demand π^* characterized by the ODE (2.19) is a well-defined real-valued C^1 -function defined on $[0, T]$, which maps into $[\pi^*(T), \pi^\infty)$ if $\pi^*(T) < \pi^\infty$, or maps into $(\pi^\infty, \pi^*(T)]$ if $\pi^*(T) > \pi^\infty$, or is equal to a constant π^∞ if $\pi^*(T) = \pi^\infty$;
- (b) f , $\frac{c^*}{X^*}$ and $\tilde{\pi}^*$ are well-defined real-valued C^1 -functions given by (2.17), (2.18) and (2.13);
- (c) The triplet $(c^*, \pi^*, \tilde{\pi}^*)$ is admissible in terms of Definition 2.17.

Then, we can verify that our candidate strategy is indeed optimal.

¹⁶The property $f \in C^1$ was established in Proposition 2.15.

Theorem 2.19. (*Verification for the finite-horizon case*). Under Assumptions 2.1, 2.7, and 2.11, the function G in (2.4) is the value function, and $(c^*, \pi^*, \tilde{\pi}^*)$ given by (2.18), (2.19), and (2.13) is the corresponding optimal strategy.

With an infinite horizon, the problem is time-independent, as discussed in Section 2.3, so all time derivatives, including G_t and $\frac{d\pi}{dt}$, vanish. The candidate optimal demand for the first stock is the constant given in Proposition 2.12. Consequently, f , $\frac{c^*}{X^*}$, and $\tilde{\pi}^*$ given by (2.17), (2.18), and (2.13) are also constant.

To extend the verification to the infinite-horizon case, it remains to show transversality ensuring that the bequest term vanishes asymptotically. The following theorem establishes this result:

Theorem 2.20. (*Verification for infinite-horizon case*). Under Assumptions 2.1, 2.7 and 2.11, π^∞ given by Proposition 2.12 is the optimal demand for the first stock if the time horizon is infinite, and the value function is given by

$$G(x) = \frac{1}{1-\gamma} x^{1-\gamma} f^\kappa$$

where f is a constant given by substituting $\pi = \pi^\infty$ into (2.17). The optimal consumption-wealth ratio and the optimal demand for the second stock are also constants and given by

$$\frac{c^\infty}{x} = \frac{\alpha}{1-\alpha} [(1-\rho^2)\gamma\sigma^2\pi^\infty - [(\eta - A(\pi^\infty)) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\pi^\infty))]]\pi^\infty, \quad (2.26)$$

$$\tilde{\pi}^\infty = \frac{\tilde{\eta} - B(\pi^\infty)}{\gamma\tilde{\sigma}^2} - \rho\frac{\sigma}{\tilde{\sigma}}\pi^\infty. \quad (2.27)$$

3 Implications for Portfolio Choice

If an investor prefers some assets over others, demand for the favored asset always increases relative to a scenario without such preferences. This is intuitive and formally shown in Proposition 2.5. We refer to this phenomenon as the preference effect. As a consequence, demand for non-favored assets is crowded out if favored and non-favored assets covary positively. Proposition 2.4 establishes this result. We call this the substitution effect. The magnitude of this effect is less obvious.

First, the investor may leave her position in the non-favored asset unchanged and simply reduce her cash holdings.¹⁷ Second, even if she reduces her position in the non-favored asset, three scenarios are possible: (i) the increase in the favored asset is exactly offset by a decrease in the non-favored asset. In this case, the risky share, defined as the sum of all risky holdings, remains unchanged. (ii) The increase in favored holdings exceeds the decrease in non-favored holdings. In this case, the risky share

¹⁷Below, we argue that this scenario occurs if the two risky assets have zero covariation. Proposition 2.5 can be extended to cover this case.

increases. (iii) The increase in favored holdings is smaller than the decrease in non-favored holdings. In this case, the risky share decreases.

In this section, we argue that all four cases are possible. In fact, there is a subtle interplay between preferences and the covariation between favored and non-favored assets. The mere existence of such preferences can lead to any of these cases. Which case materializes is determined solely by the covariation between the assets, while the magnitude of the resulting distortions depends on the strength of the preferences.

Therefore, preferences for a particular asset affect both portfolio composition and the risky share. This differs from preferences for cash or for all risky assets. In these cases, cash holdings or the risky share increase. Notice that cash and risky assets do not comove.

To formally analyze this issue, we first define the risky share as

$$\Pi^* := \pi^* + \tilde{\pi}^*,$$

which is the total proportion of wealth invested in risky assets. In our setup, we can express the share of the non-favored asset as a function of the share of the favored asset and obtain

$$\Pi^*(\pi^*(t)) = \pi^*(t) + \tilde{\pi}(\pi^*(t)),$$

where $\tilde{\pi}(\pi)$ is given by Proposition 2.4. Further, $\pi^*(t) \equiv m$ in the case without stock-specific preferences, $\pi^*(t) \in (m, \frac{1}{L_1})$ is given by Proposition 2.15 in the finite-horizon case with a preference for the first stock, and $\pi^*(t) \equiv \pi^\infty \in (m, \frac{1}{L_1})$ is given by Proposition 2.12 in the infinite-horizon case with a preference for the first stock. The agent's cash position is then given by $1 - \Pi^*(\pi^*(t))$.

To see whether the agent engages in excessive risk-taking at any $\pi^*(t) = \pi \in (m, \frac{1}{L_1})$ fixed, we differentiate $\Pi^*(\pi^*(t))$ with respect to $\pi^*(t)$ at $\pi^*(t) = \pi$. By equation (2.14), we have

$$\left. \frac{d\Pi^*}{d\pi^*} \right|_{\pi^*=\pi} = 1 + \left. \frac{d\tilde{\pi}}{d\pi} \right|_{\pi^*=\pi} = \frac{\tilde{\lambda}L_2^2(1 - L_2\tilde{\pi})^{-\gamma-1} + \tilde{\sigma}^2(1 - \rho\frac{\sigma}{\tilde{\sigma}}) + \tilde{\lambda}\hat{L}_2(\hat{L}_2 - \hat{L}_1)(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}}{\tilde{\sigma}^2 + \tilde{\lambda}L_2^2(1 - L_2\tilde{\pi})^{-\gamma-1} + \tilde{\lambda}\hat{L}_2^2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}}, \quad (3.1)$$

where we have omitted the dependence of $\tilde{\pi}$ on π for notational convenience. We first focus on the case without jumps. Then this derivative simplifies and becomes explicit

$$\left. \frac{d\Pi^*}{d\pi^*} \right|_{\pi^*=\pi} = 1 - \frac{\rho\sigma}{\tilde{\sigma}} = 1 - \beta, \quad (3.2)$$

where $\beta := \rho\sigma\tilde{\sigma}/\tilde{\sigma}^2$ is the instantaneous beta of the favored stock with respect to the non-favored stock.

This representation is powerful and allows us to distinguish two main cases:¹⁸ $\beta < 1$ and $\beta > 1$. In the first case, the risky share increases, whereas in the second case it decreases. Therefore, the risky share increases if the common risk component $\rho\sigma$ of the favored asset is smaller than the total risk $\tilde{\sigma}$ of the non-favored asset. If it is larger, the risky share decreases. The economic intuition for this result is that the agent is aware that her preference bias changes her optimal risky position. Therefore, she tries to offset this bias by adjusting her position in the non-favored asset according to the common risk component $\rho\sigma$. If the latter is high, i.e., if the favored asset is relatively risky, the preference bias can lead to an overall reduction in stock exposure. Otherwise, the response is more moderate.

In the special case of zero correlation, $\rho = 0$, the agent simply increases her position in the favored asset. Since there is no common risk component, adjusting the position in the non-favored asset provides no benefit. By contrast, if the common risk component equals the total risk of the non-favored asset, $\rho\sigma = \tilde{\sigma}$, the total risky share remains unchanged: the agent exactly offsets the increase in the favored position by an equal reduction in the non-favored position.

To isolate the effect of jumps, we consider a model without diffusive risk, $\sigma = \tilde{\sigma} = 0$. Then we obtain

$$\left. \frac{d\Pi^*}{d\pi^*} \right|_{\pi^*=\pi} = 1 - \frac{\hat{\lambda}\hat{L}_1\hat{L}_2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}}{\tilde{\lambda}L_2^2(1 - L_2\tilde{\pi})^{-\gamma-1} + \hat{\lambda}\hat{L}_2^2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}}. \quad (3.3)$$

This is an implicit representation of the derivative, since the right-hand side depends on the portfolio shares. The numerator involving $\hat{\lambda}\hat{L}_1\hat{L}_2$ now measures the size of the common risk component. Therefore, arguments similar to those above apply, and the results do not change qualitatively. In particular, the risky share remains unchanged in the homogeneous case with $\hat{L}_1 = \hat{L}_2$ and no unsystematic risk, $\tilde{\lambda} = 0$. Finally, we point out that the general case (3.1) can be expressed as a weighted average of the two polar cases (3.2) and (3.3).

4 Numerical Examples

We now calibrate our model to U.S. data and assume that an investor prefers “green” over “brown” stocks. We choose the MSCI KLD 400 Social Index as the first stock and the S&P 500 Energy Index as the second stock. The MSCI KLD 400 Social Index is a representative value-weighted U.S. equity index that includes companies with strong ESG ratings and excludes companies whose products have negative social or environmental impacts.¹⁹ It thus provides a natural proxy for an ESG portfolio in our model, in which the agent favors stocks with high ESG scores. The S&P 500 Energy Index is a value-weighted index comprising S&P 500 constituents classified as members of the GICS energy sector.²⁰ We use it as a proxy for a portfolio of brown stocks, since these firms generally have high

¹⁸The special case $\beta = 1$ is discussed below.

¹⁹<https://www.msci.com/indexes/index/700727>.

²⁰<https://www.spglobal.com/spdji/en/indices/equity/sp-500-energy-sector/>.

carbon emissions and dependence on fossil fuels and are frequent targets of ESG divestment. We obtain daily cum-dividend index data for both indices from the LSEG database for November 1994 through March 2025, a period of 30.42 years. To estimate the risk-free rate, we use monthly one-month Treasury bill rates from Kenneth French’s data library for July 1926 through March 2025. All returns are deflated using the corresponding monthly Consumer Price Index for All Urban Consumers (CPI-U) reported by the U.S. Bureau of Labor Statistics. Additional details can be found in Appendix E. Our baseline and alternative calibrations are summarized in Table 1.

For the baseline calibration, the optimal demand for the first stock is $m = 0.4705$ if the agent has no stock-specific preferences. The two lower bounds implied by Assumptions 2.7 and 2.11 are -0.0382 and -0.3201 , respectively. Clearly, the actual risk-free rate lies well above both values. The fact that these bounds are so low suggests that Assumptions 2.7 and 2.11 impose only weak restrictions.

To solve the model, we first define a function that numerically computes $\tilde{\pi}$ for any given $\pi \in \mathbb{R}$ from equation (2.13), i.e., the implicit function characterized in Proposition 2.4. We then substitute this function for $\tilde{\pi}$ in the remaining equations and use finite-difference methods or root-finding algorithms to solve the resulting system. More details can be found in Appendix C.

Table 1: Calibrations. This table summarizes the parameter values used in the baseline and alternative case. If there is no entry in the last column labeled “Alternative”, we use the same value in both cases, “Baseline” and “Alternative”.

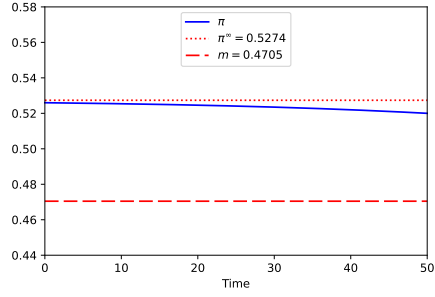
Parameter	Meaning	Baseline	Alternative
r	Risk-free rate	0.0030	
η	First stock’s excess return parameter	0.1506	
σ	First stock’s volatility	0.1822	0.3500
λ	First stock’s idiosyncratic jump intensity	0.2630	
L_1	First stock’s idiosyncratic jump size	0.0574	0.0100
$\hat{L}_1$	First stock’s co-jump size	0.0744	0.1766
$\tilde{\eta}$	Second stock’s excess return parameter	0.1799	
$\tilde{\sigma}$	Second stock’s volatility	0.2531	0.1366
$\tilde{\lambda}$	Second stock’s idiosyncratic jump intensity	0.2301	
L_2	Second stock’s idiosyncratic jump size	0.0743	0.2800
$\hat{L}_2$	Second stock’s co-jump size	0.1180	0.1441
ρ	Instantaneous diffusive correlation	0.5769	0.9000
$\hat{\lambda}$	Co-jump intensity	0.4274	0.5459
γ	Coefficient of relative risk aversion	5	
δ	Time preference rate	0.0300	
α	Consumption weight in instantaneous utility	0.9000	
$\tilde{\alpha}$	Consumption weight in bequest motive	1	
$\pi(T)$	Final share of favored stock	0.5200	0.0600
T	Time horizon	50	

As expected, Panels (a) and (c) of Figure 1 show that demand for the favored asset increases, whereas the share of the non-favored asset decreases. The overall risky share increases, as can be seen in

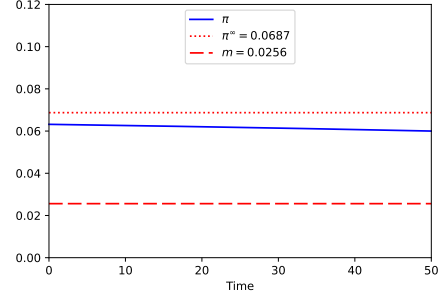
Panel (e). For our calibration, the effect of preferences weakens when we consider a finite time horizon. This is because the shares of both stocks move toward their Merton benchmark levels. By varying $\pi(T)$, one can change this result and also generate the opposite pattern.²¹

The results for the alternative calibration are depicted in Panels (b), (d), and (f). The purpose of this calibration is to show that the model can generate cases in which the total risky share decreases, as illustrated in Panel (f). The economic intuition has been explained in Section 3. As can be seen from the last column of Figure 1, the favored asset is now riskier than the non-favored asset in the sense that its volatility σ and common jump size $\hat{L}_1$ are higher. Consequently, the investor exposes herself to more common risk. To compensate for this additional exposure induced by her preference bias, she reduces her position in the non-favored asset by more than she increases her position in the favored asset, so that the total risky share is lower than without asset tastes.

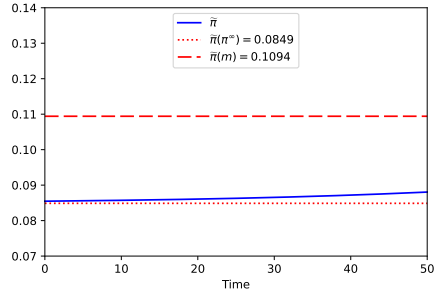
²¹The corresponding graphs are available upon request.



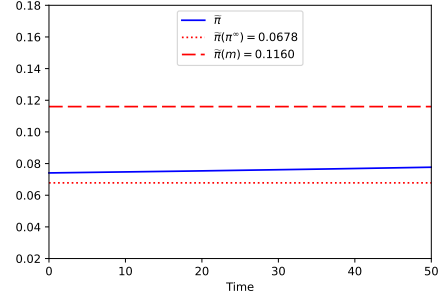
(a) Share of Favored Asset, Baseline



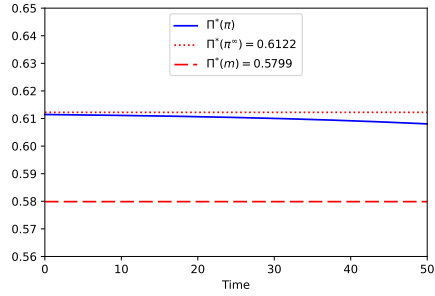
(b) Share of Favored Asset, Alternative



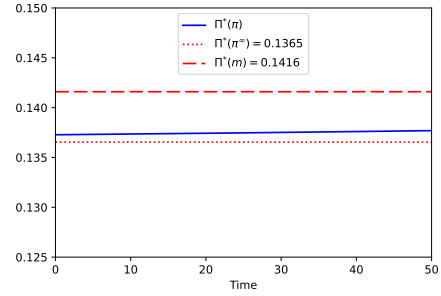
(c) Share of Non-Favored Asset, Baseline



(d) Share of Non-Favored Asset, Alternative



(e) Total Risky Share, Baseline



(f) Total Risky Share, Alternative

Figure 1: Individual Portfolio Shares and Total Risky Shares. This figure depicts the risky shares as well as the total risky share for the baseline calibration and the alternative calibration reported in Table 1.

5 Conclusion

This paper studies a dynamic consumption–portfolio choice problem with two risky stocks exposed to jump-diffusion risks. The investor derives non-pecuniary utility from holding one of the stocks. As highlighted by Fama and French (2007), the favored stock combines features of both a consumption good and a risky asset. We derive semi-analytical solutions in both finite- and infinite-horizon settings and establish their existence, uniqueness, and optimality under weak and economically plausible conditions. In both settings, the optimal demand for the favored stock always exceeds its counterpart in the benchmark model without stock-specific preferences. In the finite-horizon case, this demand is either increasing or decreasing over time, depending on the agent’s bequest motive. Whether the agent shifts her investment toward risky assets or cash depends on the sensitivity of the favored stock to the non-favored stock.

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A Proofs for the Infinite Horizon

Proof of Proposition 2.3. $H(c, \pi, \tilde{\pi})$ is strictly concave for all $(c, \pi, \tilde{\pi}) \in (0, \infty) \times (0, \frac{1}{L_1}) \times (-\infty, \frac{1}{L_2})$ and $\hat{L}_1\pi + \hat{L}_2\tilde{\pi} < 1$ if its Hessian matrix is negative definite in this domain, which is true if and only if the leading principal minors alternate in sign, i.e., if the following conditions hold

$$H_{cc} < 0, \quad \det \begin{bmatrix} H_{cc} & H_{c\pi} \\ H_{\pi c} & H_{\pi\pi} \end{bmatrix} > 0, \quad \det \begin{bmatrix} H_{cc} & H_{c\pi} & H_{c\tilde{\pi}} \\ H_{\pi c} & H_{\pi\pi} & H_{\pi\tilde{\pi}} \\ H_{\tilde{\pi}c} & H_{\tilde{\pi}\pi} & H_{\tilde{\pi}\tilde{\pi}} \end{bmatrix} < 0,$$

for all $(c, \pi, \tilde{\pi}) \in (0, \infty) \times (0, \frac{1}{L_1}) \times (-\infty, \frac{1}{L_2})$ and $\widehat{L}_1\pi + \widehat{L}_2\tilde{\pi} < 1$.

Combining with conjecture (2.4), it is straightforward to compute that

$$\begin{aligned}
H_{cc} &= u_{cc} = -\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}(x\pi)^{(1-\alpha)(1-\gamma)} < 0; \\
H_{c\pi} &= H_{\pi c} = -\alpha(1 - \alpha)(\gamma - 1)c^{\alpha(1-\gamma)-1}x^{(1-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)-1}; \\
H_{\pi\pi} &= -\gamma\sigma^2x^{1-\gamma}f^k - \gamma\lambda L_1^2x^{1-\gamma}(1 - L_1\pi)^{-\gamma-1}f^k \\
&\quad - \gamma\widehat{\lambda}\widehat{L}_1^2x^{1-\gamma}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^k \\
&\quad + (1 - \alpha)[(1 - \alpha)(\gamma - 1) + 1]c^{\alpha(1-\gamma)}x^{(1-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)-2}; \\
H_{\tilde{\pi}c} &= H_{c\tilde{\pi}} = 0; \\
H_{\tilde{\pi}\pi} &= H_{\pi\tilde{\pi}} = -\gamma\rho\sigma\tilde{\sigma}x^{1-\gamma}f^k - \gamma\widehat{\lambda}\widehat{L}_1\widehat{L}_2x^{1-\gamma}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^k; \\
H_{\tilde{\pi}\tilde{\pi}} &= -\gamma\tilde{\sigma}^2x^{1-\gamma}f^k - \gamma\tilde{\lambda}\widehat{L}_2^2x^{1-\gamma}(1 - L_2\tilde{\pi})^{-\gamma-1}f^k \\
&\quad - \gamma\widehat{\lambda}\widehat{L}_2^2x^{1-\gamma}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^k.
\end{aligned}$$

Therefore, for all $(c, \pi, \tilde{\pi}) \in (0, \infty) \times (0, \frac{1}{L_1}) \times (-\infty, \frac{1}{L_2})$ and $\widehat{L}_1\pi + \widehat{L}_2\tilde{\pi} < 1$, we have

$$\begin{aligned}
\det \begin{bmatrix} H_{cc} & H_{c\pi} \\ H_{\pi c} & H_{\pi\pi} \end{bmatrix} &= H_{cc}H_{\pi\pi} - H_{c\pi}^2 \\
&= \gamma\sigma^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(2-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}f^k \\
&\quad + \gamma\lambda L_1^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(2-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - L_1\pi)^{-\gamma-1}f^k \\
&\quad + \gamma\widehat{\lambda}\widehat{L}_1^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(2-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^k \\
&\quad + \gamma\alpha(1 - \alpha)c^{2\alpha(1-\gamma)-2}x^{2(1-\alpha)(1-\gamma)}\pi^{2(1-\alpha)(1-\gamma)-2} > 0,
\end{aligned}$$

and also

$$\begin{aligned}
\det \begin{bmatrix} H_{cc} & H_{c\pi} & H_{c\tilde{\pi}} \\ H_{\pi c} & H_{\pi\pi} & H_{\pi\tilde{\pi}} \\ H_{\tilde{\pi}c} & H_{\tilde{\pi}\pi} & H_{\tilde{\pi}\tilde{\pi}} \end{bmatrix} &= H_{\tilde{\pi}c} \det \begin{bmatrix} H_{c\pi} & H_{c\tilde{\pi}} \\ H_{\pi\pi} & H_{\pi\tilde{\pi}} \end{bmatrix} - H_{\tilde{\pi}\pi} \det \begin{bmatrix} H_{cc} & H_{c\tilde{\pi}} \\ H_{\pi c} & H_{\pi\tilde{\pi}} \end{bmatrix} + H_{\tilde{\pi}\tilde{\pi}} \det \begin{bmatrix} H_{cc} & H_{c\pi} \\ H_{\pi c} & H_{\pi\pi} \end{bmatrix} \\
&= -\gamma^2\widehat{\lambda}(\widehat{\sigma}\widehat{L}_1 - \sigma\widehat{L}_2)^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^{2k} \\
&\quad - 2(1 - \rho)\gamma^2\widehat{\lambda}\widehat{L}_1\widehat{L}_2\sigma\tilde{\sigma}\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^{2k} \\
&\quad - (1 - \rho^2)\gamma^2\sigma^2\tilde{\sigma}^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}f^{2k} \\
&\quad - \gamma^2\lambda L_1^2\tilde{\sigma}^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - L_1\pi)^{-\gamma-1}f^{2k} \\
&\quad - \gamma^2\tilde{\lambda}\widehat{L}_2^2\sigma^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - L_2\tilde{\pi})^{-\gamma-1}f^{2k} \\
&\quad - \gamma^2\widehat{\lambda}\widehat{L}_1^2L_2^2\tilde{\sigma}^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - L_1\pi)^{-\gamma-1}(1 - L_2\tilde{\pi})^{-\gamma-1}f^{2k} \\
&\quad - \gamma^2\widehat{\lambda}\widehat{L}_1^2\widehat{L}_2^2\tilde{\sigma}^2\alpha[\alpha(\gamma - 1) + 1]c^{\alpha(1-\gamma)-2}x^{(3-\alpha)(1-\gamma)}\pi^{(1-\alpha)(1-\gamma)}(1 - L_1\pi)^{-\gamma-1}(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})^{-\gamma-1}f^{2k}
\end{aligned}$$

$$\begin{aligned}
& -\gamma^2 \tilde{\lambda} \hat{\lambda} L_2^2 \hat{L}_1^2 \tilde{\sigma}^2 \alpha [\alpha(\gamma-1) + 1] c^{\alpha(1-\gamma)-2} x^{(3-\alpha)(1-\gamma)} \pi^{(1-\alpha)(1-\gamma)} (1 - L_2 \tilde{\pi})^{-\gamma-1} (1 - \hat{L}_1 \pi - \hat{L}_2 \tilde{\pi})^{-\gamma-1} f^{2k} \\
& -\gamma^2 \tilde{\sigma}^2 \alpha (1-\alpha) c^{2\alpha(1-\gamma)-2} x^{(3-2\alpha)(1-\gamma)} \pi^{2(1-\alpha)(1-\gamma)-2} f^k \\
& -\gamma^2 \tilde{\lambda} L_2^2 \alpha (1-\alpha) c^{2\alpha(1-\gamma)-2} x^{(3-2\alpha)(1-\gamma)} \pi^{2(1-\alpha)(1-\gamma)-2} (1 - L_2 \tilde{\pi})^{-\gamma-1} f^k \\
& -\gamma^2 \tilde{\lambda} \hat{L}_2^2 \alpha (1-\alpha) c^{2\alpha(1-\gamma)-2} x^{(3-2\alpha)(1-\gamma)} \pi^{2(1-\alpha)(1-\gamma)-2} (1 - \hat{L}_1 \pi - \hat{L}_2 \tilde{\pi})^{-\gamma-1} f^k < 0.
\end{aligned}$$

Thus, all the conditions are satisfied. $\square$

Proof of Proposition 2.4. First, for a fixed $\pi \in \mathbb{R}$ equation (2.13) is nonlinear with respect to $\tilde{\pi}$:

$$\tilde{\pi} + \frac{\bar{B}(\pi, \tilde{\pi})}{\gamma \tilde{\sigma}^2} = \frac{\tilde{\eta}}{\gamma \tilde{\sigma}^2} - \rho \frac{\sigma}{\tilde{\sigma}} \pi, \quad (\text{A.1})$$

where $\bar{B}(\pi, \tilde{\pi}) = \tilde{\lambda} L_2 (1 - L_2 \tilde{\pi})^{-\gamma} + \hat{\lambda} \hat{L}_2 (1 - \hat{L}_1 \pi - \hat{L}_2 \tilde{\pi})^{-\gamma}$. Then it is clear that $\bar{B} \rightarrow \infty$ as $\tilde{\pi} \nearrow \min(\frac{1}{L_2}, \frac{1-\hat{L}_1 \pi}{\hat{L}_2})$ and that $\bar{B} \rightarrow 0$ as $\tilde{\pi} \rightarrow -\infty$. Furthermore, it is easy to check that $\bar{B}$ is continuous and monotonically increasing with respect to $\tilde{\pi}$ given π fixed. Therefore, the left-hand side of (A.1) is also continuous and monotonically (strictly) increasing on $\tilde{\pi} \in (-\infty, \min(\frac{1}{L_2}, \frac{1-\hat{L}_1 \pi}{\hat{L}_2}))$, which goes through all the values in $\mathbb{R}$. Hence, given that $\pi \in \mathbb{R}$ is only a fixed real number, by the intermediate value theorem, there must be one unique solution $\tilde{\pi} \in (-\infty, \min(\frac{1}{L_2}, \frac{1-\hat{L}_1 \pi}{\hat{L}_2}))$ that solves (A.1).

Second, we define

$$g(\pi, \tilde{\pi}) := \tilde{\pi} + \frac{\bar{B}(\pi, \tilde{\pi}) - \tilde{\eta}}{\gamma \tilde{\sigma}^2} + \rho \frac{\sigma}{\tilde{\sigma}} \pi.$$

Then, we have shown that for any fixed π_0 , a unique $\tilde{\pi}_0$ exists that solves (A.1), i.e., the pair $(\pi_0, \tilde{\pi}_0)$ satisfies $g(\pi_0, \tilde{\pi}_0) = 0$. Given that the partial derivative of g with respect to $\tilde{\pi}$ at $(\pi_0, \tilde{\pi}_0)$ is

$$\left. \frac{\partial g}{\partial \tilde{\pi}} \right|_{(\pi_0, \tilde{\pi}_0)} = 1 + \frac{1}{\tilde{\sigma}^2} [\tilde{\lambda} L_2^2 (1 - L_2 \tilde{\pi}_0)^{-\gamma-1} + \hat{\lambda} \hat{L}_2^2 (1 - \hat{L}_1 \pi_0 - \hat{L}_2 \tilde{\pi}_0)^{-\gamma-1}],$$

which is always strictly positive. Therefore, by the implicit function theorem, there exists an open set $U \subset \mathbb{R}$ that contains π_0 such that there is a unique continuously differentiable function $\tilde{\pi}(\pi)$ defined on U that satisfies $g(\pi, \tilde{\pi}(\pi)) = 0$ on U and its derivative is given by:

$$\frac{d\tilde{\pi}}{d\pi} = -\frac{\partial g / \partial \pi}{\partial g / \partial \tilde{\pi}} = -\frac{\rho \sigma \tilde{\sigma} + \hat{\lambda} \hat{L}_1 \hat{L}_2 (1 - \hat{L}_1 \pi - \hat{L}_2 \tilde{\pi})^{-\gamma-1}}{\tilde{\sigma}^2 + \tilde{\lambda} L_2^2 (1 - L_2 \tilde{\pi})^{-\gamma-1} + \hat{\lambda} \hat{L}_2^2 (1 - \hat{L}_1 \pi - \hat{L}_2 \tilde{\pi})^{-\gamma-1}} < 0,$$

where the inequality holds since $\rho \geq 0$ by Assumption 2.1 (b). Finally, since the above argument applies to every $\pi \in \mathbb{R}$, it is clear that $\tilde{\pi}(\pi)$ is continuously differentiable everywhere (uniqueness and differentiability of $\tilde{\pi}(\pi)$ on an open set around π_0 imply the uniqueness and differentiability on $\mathbb{R}$ since we can always find overlapping open sets for arbitrarily close π 's). $\square$

Proof of Proposition 2.5. (a) Notice that the right-hand side of equation (2.12) requires $\pi^*(t) > 0$. Besides, given Assumption 2.2 (a), we have $f(t) > 0$. Therefore, this implies that (2.12)'s left-hand side is also positive, i.e., $\pi^*(t) > \frac{1}{(1-\rho^2)\gamma\sigma^2}[(\eta - A(\pi^*(t))) - \rho\frac{\sigma}{\bar{\sigma}}(\tilde{\eta} - B(\pi^*(t)))]$.

(b) Notice that (2.13) and (F.5) are of exactly the same functional form. Thus Proposition 2.4 also applies to the standard problem discussed in Appendix F. Therefore, both models (Section 2 and Appendix F) share the same implicit function $\tilde{\pi}(\pi)$ derived from $\tilde{\pi}$'s first-order conditions, which further leads to the same functions of $A(\pi)$ and $B(\pi)$ by substituting the implicit function $\tilde{\pi}(\pi)$ into $\bar{A}(\pi, \tilde{\pi})$ and $\bar{B}(\pi, \tilde{\pi})$ in equation (F.2). Hence, since $\pi = m$ is defined as the solution to the problem without stock-specific preferences, by equation (F.2), we obtain that

$$m = \frac{1}{(1-\rho^2)\gamma\sigma^2}[(\eta - A(m)) - \rho\frac{\sigma}{\bar{\sigma}}(\tilde{\eta} - B(m))]. \quad (\text{A.2})$$

Besides, using (2.14), we calculate the following derivative:

$$\begin{aligned} \frac{dA}{d\pi} - \rho\frac{\sigma}{\bar{\sigma}}\frac{dB}{d\pi} &= \frac{\gamma}{\tilde{\sigma}^2 + \tilde{\lambda}L_2^2(1 - L_2\tilde{\pi})^{-\gamma-1} + \hat{\lambda}\hat{L}_2^2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1}} \\ &\quad \{ \tilde{\sigma}^2\lambda L_1^2(1 - L_1\pi)^{-\gamma-1} + \rho^2\sigma^2\tilde{\lambda}L_2^2(1 - L_2\tilde{\pi})^{-\gamma-1} \\ &\quad + \tilde{\sigma}^2\hat{\lambda}[\hat{L}_1 - \rho\frac{\sigma}{\bar{\sigma}}\hat{L}_2]^2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1} \\ &\quad + \lambda\tilde{\lambda}L_1^2L_2^2(1 - L_1\pi)^{-\gamma-1}(1 - L_2\tilde{\pi})^{-\gamma-1} \\ &\quad + \lambda\hat{\lambda}L_1^2\hat{L}_2^2(1 - L_1\pi)^{-\gamma-1}(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1} \\ &\quad + \tilde{\lambda}\hat{\lambda}L_2^2\hat{L}_1^2(1 - L_2\tilde{\pi})^{-\gamma-1}(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma-1} \} > 0. \end{aligned} \quad (\text{A.3})$$

Hence, if we replace m by $\pi^*(t) \leq m$ in equation (A.2), the left-hand side of (A.2) will be smaller or equal to m , whereas its right-hand side will be greater or equal to $\frac{1}{(1-\rho^2)\gamma\sigma^2}[(\eta - A(m)) - \rho\frac{\sigma}{\bar{\sigma}}(\tilde{\eta} - B(m))]$:

$$\pi^*(t) \leq m \quad \Rightarrow \quad \pi^*(t) \leq \frac{1}{(1-\rho^2)\gamma\sigma^2}[(\eta - A(\pi^*(t))) - \rho\frac{\sigma}{\bar{\sigma}}(\tilde{\eta} - B(\pi^*(t)))],$$

which immediately implies the contrapositive also holds, i.e., (a) implies $\pi^*(t) > m$.

(c) Observe that the left-hand side of equation (2.12) contains nonlinear terms with three different bases, i.e., $1 - L_1\pi^*(t)$, $1 - L_2\tilde{\pi}(\pi^*(t))$ and $1 - \hat{L}_1\pi^*(t) - \hat{L}_2\tilde{\pi}(\pi^*(t))$, all of which are raised to negative powers. Therefore, one natural requirement is that all of these terms must be positive. However, by Proposition 2.4, we have shown that for every $\pi \in \mathbb{R}$ fixed, $\tilde{\pi}(\pi) < \min(\frac{1}{L_2}, \frac{1-\hat{L}_1\pi}{\hat{L}_2})$. Therefore, $1 - L_2\tilde{\pi}(\pi) > 0$ and $1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi}(\pi) > 0$ must hold for every $\pi \in \mathbb{R}$. Hence, the only inequality that needs to be imposed is $1 - L_1\pi^*(t)$, i.e., $\pi^*(t) < \frac{1}{L_1}$. $\square$

Proof of Proposition 2.6. Given equation (2.17) which expresses f with π , the time derivative of f can be obtained using the chain rule

$$\frac{df}{dt} = -f \frac{(1 - \rho^2)\gamma\sigma^2 + \frac{\gamma}{\kappa}[(1 - \rho^2)\gamma\sigma^2\pi - [(\eta - A) - \rho\frac{\sigma}{\sigma}(\tilde{\eta} - B)]]\pi^{-1} + [\frac{dA}{d\pi} - \rho\frac{\sigma}{\sigma}\frac{dB}{d\pi}]}{(1 - \rho^2)\gamma\sigma^2\pi - [(\eta - A) - \rho\frac{\sigma}{\sigma}(\tilde{\eta} - B)]} \frac{d\pi}{dt},$$

where we divide both the numerator and denominator in the fraction by $\pi^{\frac{\gamma}{\kappa}}$ since π is never zero by Proposition 2.5 (b) and Assumption 2.1 (c). Substituting this equation into the reduced HJB equation (2.16) yields the result. $\square$

Proof of Proposition 2.8. Under Assumptions 2.1 and 2.2, it is straightforward to see that $\gamma - 1 > 0$ and $\kappa > 0$. Besides, given these assumptions and by Proposition 2.5, $\pi > \frac{1}{(1 - \rho^2)\gamma\sigma^2}[(\eta - A) - \rho\frac{\sigma}{\sigma}(\tilde{\eta} - B)]$ and $\pi > m > 0$ hold. Furthermore, by (A.3) we have proven that $\frac{dA}{d\pi} - \rho\frac{\sigma}{\sigma}\frac{dB}{d\pi} > 0$. Thus the result follows. $\square$

Proof of Proposition 2.9. The range of $\pi(T)$ is proven by Proposition 2.5 since $t = T$ is only a special case for $\pi(t)$. Therefore, we only need to prove the existence and uniqueness of $\pi(T)$ on $(m, \frac{1}{L_1})$.

Existence and uniqueness: Notice that by equation (A.3), the left-hand side of (2.20) is monotonically increasing with respect to $\pi(T)$. Moreover, due to (A.2), it is obvious that the left-hand side increases from 0. In addition, note that as $\pi(T) \nearrow \frac{1}{L_1}$, the left-hand side goes to infinity, because $A(\pi(T))$ involves the term $(1 - L_1\pi(T))^{-\gamma}$ which will then go to infinity, whereas all the other terms will stay finite. In particular, the finiteness of $B(\pi(T))$ (and thus also of the term $(1 - \hat{L}_1\pi(T) - \hat{L}_2\tilde{\pi}(\pi(T)))^{-\gamma}$ in $A(\pi(T))$) is given by rearranging equation (2.13):

$$|B(\pi(T))| = |\tilde{\eta} - \gamma\tilde{\sigma}^2\tilde{\pi}(\pi(T)) - \gamma\rho\sigma\tilde{\sigma}\pi(T)| \leq \tilde{\eta} + \gamma\tilde{\sigma}^2|\tilde{\pi}(\pi(T))| + \gamma\rho\sigma\tilde{\sigma}|\pi(T)| < \infty,$$

which holds for all $\pi(T) \in [m, \frac{1}{L_1}]$, where the finiteness of $\tilde{\pi}$ is guaranteed by Proposition 2.4.

On the other hand, it is straightforward to see that the right-hand side of (2.20) is monotonically decreasing from $\omega^{-\frac{1}{\kappa}}\frac{1-\alpha}{\alpha}\alpha^{\frac{1}{\kappa}}m^{-\frac{\kappa\tilde{\alpha}}{\kappa}} > 0$ to $\omega^{-\frac{1}{\kappa}}\frac{1-\alpha}{\alpha}\alpha^{\frac{1}{\kappa}}(\frac{1}{L_1})^{-\frac{\kappa\tilde{\alpha}}{\kappa}} > 0$ as $\pi(T)$ moves from m to $\frac{1}{L_1}$. Therefore, by the intermediate value theorem, there must exist a unique $\pi(T)$ that solves (2.20). $\square$

Proof of Lemma 2.10. The continuity and differentiability of P directly follows from its functional form, given the fact that $\tilde{\pi}(\pi)$ is a well-defined, continuous and differentiable function on $[m, \frac{1}{L_1}]$ due to Proposition 2.4.

Next, we show that P is finite on $[m, \frac{1}{L_1}]$. For any $\pi \in (-\infty, \frac{1}{L_1}]$ fixed, the B function is always finite (i.e., will not go to infinity), since by rearranging (2.13)

$$|B(\pi)| = |\tilde{\eta} - \gamma\tilde{\sigma}^2\tilde{\pi}(\pi) - \gamma\rho\sigma\tilde{\sigma}\pi| \leq \tilde{\eta} + \gamma\tilde{\sigma}^2|\tilde{\pi}(\pi)| + \gamma\rho\sigma\tilde{\sigma}|\pi| < \infty,$$

where the finiteness of $\tilde{\pi}$ is guaranteed by Proposition 2.4. This also implies that the last two terms in the function D , i.e., $(1 - L_2\tilde{\pi}(\pi))^{1-\gamma}$ and $(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi}(\pi))^{1-\gamma}$ is finite for all $\pi \in (-\infty, \frac{1}{L_1}]$. Besides, for any $\pi \in (-\infty, \frac{1}{L_1})$ fixed, it is obvious that $(1 - L_1\pi)^{1-\gamma}$ and $(1 - L_1\pi)^{-\gamma}$ are finite. Therefore, the functions A , B and D are all finite on $\pi \in (-\infty, \frac{1}{L_1})$, and thus the function P is well-defined. Finally, as $\pi \nearrow \frac{1}{L_1}$ the only terms in P that can explode are the terms involving $1 - L_1\pi$, i.e., $-\frac{D(\pi)}{\gamma-1}$ and $\tilde{K}\pi A(\pi)$. Notice that $-\frac{1}{\gamma-1} < 0$ and $\tilde{K} < 0$ by Assumption 2.1 (a), and that $\pi > 0$ as $\pi \nearrow \frac{1}{L_1}$, it is clear that $P(\pi) \rightarrow -\infty$ as $\pi \nearrow \frac{1}{L_1}$. $\square$

Proof of Proposition 2.12. By Lemma 2.10 and given that $P(m) > 0$ due to Assumption 2.7, we already know that P must have at least one real-valued zero on $(m, \frac{1}{L_1})$. Therefore, we first define the set of P 's zero(s) on this interval as $\Pi_0 := \{\pi_0 \in (m, \frac{1}{L_1}) | P(\pi_0) = 0\}$. Since $\Pi_0 \neq \emptyset$, we can further define $\underline{\pi}_0 := \inf \Pi_0$ and $\bar{\pi}_0 := \sup \Pi_0$. By definition, it is clear that $m \leq \underline{\pi}_0 \leq \bar{\pi}_0 \leq \frac{1}{L_1}$. Next, we proceed to prove the two claims of the proposition:

(a) Suppose $\pi_0 \in \Pi_0$. Differentiating P at π_0 gives

$$P'(\pi_0) = 2\hat{K}\pi_0 + (\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta})(1 - \tilde{K}) - (1 - \tilde{K})[A(\pi_0) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi_0)] + \tilde{K} \left(\frac{dA}{d\pi} \Big|_{\pi_0} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi} \Big|_{\pi_0} \right) \pi_0.$$

On the other hand, since π_0 is a zero of the function P , by definition of P it holds that

$$P(\pi_0) = \hat{K}\pi_0^2 + (\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta})(1 - \tilde{K})\pi_0 + \tilde{K}[A(\pi_0) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi_0)]\pi_0 + r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} - \frac{D(\pi_0)}{\gamma-1} - \frac{B(\pi_0)^2}{2\gamma\tilde{\sigma}^2} = 0. \quad (\text{A.4})$$

Observing the above two equations, it is clear that by rearranging and collecting the terms in equation (A.4), we can rewrite this equation in terms of $P'(\pi_0)$:

$$\begin{aligned} & \underbrace{\{2\hat{K}\pi_0 + (\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta})(1 - \tilde{K}) - (1 - \tilde{K})[A(\pi_0) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi_0)]\}}_{=P'(\pi_0) - \tilde{K} \left(\frac{dA}{d\pi} \Big|_{\pi_0} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi} \Big|_{\pi_0} \right) \pi_0} \pi_0 - \hat{K}\pi_0^2 + [A(\pi_0) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi_0)]\pi_0 \\ & + r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} - \frac{D(\pi_0)}{\gamma-1} - \frac{B(\pi_0)^2}{2\gamma\tilde{\sigma}^2} = 0, \end{aligned}$$

or, equivalently

$$\begin{aligned} P'(\pi_0) = & \frac{1}{\pi_0} \left\{ \frac{D(\pi_0)}{\gamma-1} + \frac{B(\pi_0)^2}{2\gamma\tilde{\sigma}^2} - \left[A(\pi_0) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi_0) \right] \pi_0 + \hat{K}\pi_0^2 - r - \frac{\delta}{\gamma-1} - \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} \right\} \\ & + \tilde{K} \left(\frac{dA}{d\pi} \Big|_{\pi_0} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi} \Big|_{\pi_0} \right) \pi_0. \end{aligned}$$

Notice that we have shown in equation (A.3) that $\frac{dA}{d\pi} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi} > 0$ must always hold on $(m, \frac{1}{L_1})$. Besides, $\pi_0 > m > 0$ holds due to Assumption 2.1 (c). Hence, given that $\tilde{K} < 0$, the second term on the right-hand side must be strictly negative.

Moreover, by differentiating the non-polynomial terms within the first term at any $\pi \in [m, \frac{1}{L_1})$, we obtain after tedious calculations

$$\frac{d}{d\pi} \left[\frac{D(\pi)}{\gamma-1} + \frac{B(\pi)^2}{2\gamma\tilde{\sigma}^2} - \left(A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi) \right) \pi \right] = - \left(\frac{dA}{d\pi} - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi} \right) \pi < 0,$$

where the inequality holds because of (A.3) and Assumption 2.1 (c). Therefore, these terms as a whole are strictly decreasing on $[m, \frac{1}{L_1})$.

Furthermore, since $\hat{K} < 0$, the term $\hat{K}\pi^2$ is also strictly decreasing on $[m, \frac{1}{L_1}) \subset \mathbb{R}_+$. Thus, we can conclude that the function $[\frac{D(\pi)}{\gamma-1} + \frac{B(\pi)^2}{2\gamma\tilde{\sigma}^2} - [A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi)]\pi + \hat{K}\pi^2 - r - \frac{\delta}{\gamma-1} - \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2}]$ is monotonically decreasing on $[m, \frac{1}{L_1})$. However, by Assumption 2.11, it is obvious that this function is nonpositive at $\pi = m$, which implies that for $\pi = \pi_0 \in (m, \frac{1}{L_1})$, the value of this function must be strictly negative. Finally, given that $\pi_0 > m > 0$, it must hold that $P'(\pi_0) < 0$.

(b) The aim is to show that the set Π_0 is a singleton. We start by showing that $P(\underline{\pi}_0) = P(\bar{\pi}_0) = 0$, and that $\underline{\pi}_0 > m$ and $\bar{\pi}_0 < \frac{1}{L_1}$. First, suppose $\bar{\pi}_0 = \frac{1}{L_1}$. Since P is not defined at $\pi = \frac{1}{L_1}$, this implies that there must be a sequence of zeros that converges towards $\frac{1}{L_1}$ from below. However, since P is continuous on $[m, \frac{1}{L_1})$ and $\lim_{\pi \nearrow \frac{1}{L_1}} P(\pi) = -\infty$ by Lemma 2.10, for very small $\varepsilon > 0$ it must hold that $P(\pi) < 0$ for all $\pi \in (\frac{1}{L_1} - \varepsilon, \frac{1}{L_1})$, which excludes the possibility of the existence of such a convergent sequence. Thus, this is a contradiction and $\bar{\pi}_0 < \frac{1}{L_1}$ should hold.

Therefore, it must hold that $m \leq \underline{\pi}_0 \leq \bar{\pi}_0 < \frac{1}{L_1}$, and thus the function P is well-defined and continuous on $[\underline{\pi}_0, \bar{\pi}_0]$ by Lemma 2.10. By definition of $\underline{\pi}_0$, either of the following two cases must hold true: (i) $P(\underline{\pi}_0) = 0$; (ii) $P(\underline{\pi}_0) \neq 0$ but for any $\varepsilon > 0$, there exists $\pi_0 \in (\underline{\pi}_0, \underline{\pi}_0 + \varepsilon)$ such that $P(\pi_0) = 0$. However, the latter should not be the case:

Suppose $P(\underline{\pi}_0) \neq 0$. Then we take any $\varepsilon_1 > 0$, and pick the zero $\pi_{0,1} \in (\underline{\pi}_0, \underline{\pi}_0 + \varepsilon_1)$. Next, we define $\varepsilon_2 := \min(\pi_{0,1}, \frac{1}{2})$ and pick the zero $\pi_{0,2} \in (\underline{\pi}_0, \underline{\pi}_0 + \varepsilon_2)$. Continuing by defining $\varepsilon_n := \min(\pi_{0,n-1}, \frac{1}{n})$ and picking $\pi_{0,n} \in (\underline{\pi}_0, \underline{\pi}_0 + \varepsilon_n)$, it is obvious that $\lim_{n \rightarrow \infty} \pi_{0,n} = \underline{\pi}_0$ with $P(\pi_{0,n}) = 0$ for all $n \in \mathbb{N}_+$. This implies that $\lim_{n \rightarrow \infty} P(\pi_{0,n}) = 0 \neq P(\underline{\pi}_0)$, which contradicts the continuity of P .

The case of $\bar{\pi}_0$ can be proven in a symmetric way. Therefore, it holds that $P(\underline{\pi}_0) = P(\bar{\pi}_0) = 0$, which further implies $\underline{\pi}_0 > m$ since $P(m) > 0$ by Assumption 2.7.

Next, we show that Π_0 must be a singleton, i.e., $\underline{\pi}_0 = \bar{\pi}_0$. We prove by contradiction. Suppose $\underline{\pi}_0 < \bar{\pi}_0$, i.e., there is more than one zero of P on $(m, \frac{1}{L_1})$. We distinguish between two cases:

If Π_0 is dense in $[\underline{\pi}_0, \bar{\pi}_0]$, then for any $\pi \in [\underline{\pi}_0, \bar{\pi}_0]$ and any $\varepsilon > 0$, there exists at least one $\pi_0 \in (\pi - \varepsilon, \pi + \varepsilon) \cap [\underline{\pi}_0, \bar{\pi}_0]$ such that $P(\pi_0) = 0$. Hence, we can first pick an arbitrary interior point $\pi \in (\underline{\pi}_0, \bar{\pi}_0)$ and any $\varepsilon_1 \in (0, \min(\pi - \underline{\pi}_0, \bar{\pi}_0 - \pi))$ to find an interior zero $\pi_0 \in (\pi - \varepsilon_1, \pi + \varepsilon_1) \subset [\underline{\pi}_0, \bar{\pi}_0]$. Then, due to the fact that $P'(\pi_0) < 0$ as proven by (a), there exists $\delta_1 \in (0, \bar{\pi}_0 - \pi_0)$ such that $P(\pi) < 0$ for all $\pi \in (\pi_0, \pi_0 + \delta_1)$. On the other hand, since $\pi_0 + \frac{\delta_1}{2} \in (\underline{\pi}_0, \bar{\pi}_0)$, for $\varepsilon_2 := \frac{\delta_1}{2}$, there must exist

another zero $\pi'_0 \in (\pi_0 + \frac{\delta_1}{2} - \varepsilon_2, \pi_0 + \frac{\delta_1}{2} + \varepsilon_2) = (\pi_0, \pi_0 + \delta_1)$. Again, by the claim in (a), for π'_0 there exists $\delta_2 \in (0, \pi'_0 - \pi_0)$ such that $P(\pi) > 0$ for all $\pi \in (\pi'_0 - \delta_2, \pi_0)$. However, this suggests that on the interval $(\max(\pi'_0 - \delta_2, \pi_0), \pi'_0)$ the function P must be both positive and negative, which is a contradiction.

If Π_0 is not dense in $[\pi_0, \bar{\pi}_0]$, then there exist $\check{\pi} \in (\pi_0, \bar{\pi}_0)$ and $\check{\varepsilon} \in (0, \min(\check{\pi} - \pi_0, \bar{\pi}_0 - \check{\pi}))$ such that P has no zero on $(\check{\pi} - \check{\varepsilon}, \check{\pi} + \check{\varepsilon})$. Note that it is unlikely that $\check{\pi} = \pi_0$ or $\check{\pi} = \bar{\pi}_0$ because we have shown that both of them are zeros. Then define $\check{\pi}_0 := \sup\{\pi_0 \in [\pi_0, \check{\pi} - \check{\varepsilon}] | P(\pi_0) = 0\}$ and $\check{\bar{\pi}}_0 := \inf\{\pi_0 \in [\check{\pi} + \check{\varepsilon}, \bar{\pi}_0] | P(\pi_0) = 0\}$. Due to the same reasoning as $\bar{\pi}_0$ and π_0 , it must hold that $P(\check{\pi}_0) = P(\check{\bar{\pi}}_0) = 0$. Moreover, by definition, there should be no zero of P on $(\check{\pi}_0, \check{\bar{\pi}}_0)$. However, due to the claim in (a), there must exist $\check{\delta}_1, \check{\delta}_2 \in (0, \check{\bar{\pi}}_0 - \check{\pi}_0)$ such that $P(\pi) < 0$ for all $\pi \in (\check{\pi}_0, \check{\pi}_0 + \check{\delta}_1)$ and $P(\pi) > 0$ for all $\pi \in (\check{\bar{\pi}}_0 - \check{\delta}_2, \check{\bar{\pi}}_0)$. Hence, by the intermediate value theorem and again the fact that P is continuous on $[\check{\pi}_0, \check{\bar{\pi}}_0]$, there must be another zero on $(\check{\pi}_0, \check{\bar{\pi}}_0)$, which is a contradiction. Therefore, $\pi_0 = \bar{\pi}_0$ and thus P only has one unique zero on $(m, \frac{1}{L_1})$. $\square$

Proof of Corollary 2.13. Notice that by Proposition 2.12 (b), π^∞ is the unique zero of P on $(m, \frac{1}{L_1})$. Therefore, for any $\pi \in (m, \pi^\infty)$, we either have $P(\pi) > 0$ or $P(\pi) < 0$. However, since $P(m) > 0$ by Assumption 2.7, $P(\pi) < 0$ is impossible because in that case the intermediate value theorem would imply that there must exist another zero of P between m and π , which contradicts Proposition 2.12 (b). The same reasoning also applies to the interval $(\pi^\infty, \frac{1}{L_1})$. Given Lemma 2.10, we have that $P(\pi)$ must be strictly negative when π approaches $\frac{1}{L_1}$ from below. Thus, by symmetric argument we must have that P is strictly negative on the entire interval $(\pi^\infty, \frac{1}{L_1})$. $\square$

B Proofs for the Finite Horizon

Proof of Proposition 2.14. First, given that $\pi \in [m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2]$, by similar argument to the proof of Proposition 2.5 (b), it is clear that $\pi > m$ implies $\pi > \frac{(\eta-A)-\rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta}-B)}{(1-\rho^2)\gamma\sigma^2}$. In addition, note that $m > 0$ by Assumption 2.1 (c). Therefore, the integrand of the function $\mathcal{G}$ only has one singularity at $\pi = \pi^\infty$ on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2]$. Furthermore, by Proposition 2.4 and given that $\bar{A}(\pi, \tilde{\pi})$ and $\bar{B}(\pi, \tilde{\pi})$ are two continuous mappings of π and $\tilde{\pi}$, the integrand of $\mathcal{G}$ must also be continuous on $\pi \in [m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$. Therefore, by the first fundamental theorem of calculus, for arbitrarily small $\varepsilon_3, \varepsilon_4 > 0$, the antiderivative of the integrand must exist on the closed intervals $[m + \varepsilon_1, \pi^\infty - \varepsilon_3]$ and $[\pi^\infty + \varepsilon_4, \frac{1}{L_1} - \varepsilon_2]$,

which is uniformly continuous on both intervals, differentiable on the open intervals $(m + \varepsilon_1, \pi^\infty - \varepsilon_3)$ and $(\pi^\infty + \varepsilon_4, \frac{1}{L_1} - \varepsilon_2)$, and has the following representation:

$$\mathcal{G}(\pi) := \begin{cases} \int_{m+\varepsilon_1}^{\pi} \frac{x + \frac{\gamma}{\kappa} [x - \frac{(\eta-A(x)) - \rho \frac{\sigma}{\sigma} (\tilde{\eta}-B(x))}{(1-\rho^2)\gamma\sigma^2}] + \frac{x}{(1-\rho^2)\gamma\sigma^2} [\frac{dA}{dx}(x) - \rho \frac{\sigma}{\sigma} \frac{dB}{dx}(x)]}{P(x)x[x - \frac{(\eta-A(x)) - \rho \frac{\sigma}{\sigma} (\tilde{\eta}-B(x))}{(1-\rho^2)\gamma\sigma^2}]} dx, & \text{if } \pi \in [m + \varepsilon_1, \pi^\infty - \varepsilon_3], \\ - \int_{\frac{1}{L_1} - \varepsilon_2}^{\pi} \frac{x + \frac{\gamma}{\kappa} [x - \frac{(\eta-A(x)) - \rho \frac{\sigma}{\sigma} (\tilde{\eta}-B(x))}{(1-\rho^2)\gamma\sigma^2}] + \frac{x}{(1-\rho^2)\gamma\sigma^2} [\frac{dA}{dx}(x) - \rho \frac{\sigma}{\sigma} \frac{dB}{dx}(x)]}{P(x)x[x - \frac{(\eta-A(x)) - \rho \frac{\sigma}{\sigma} (\tilde{\eta}-B(x))}{(1-\rho^2)\gamma\sigma^2}]} dx, & \text{if } \pi \in [\pi^\infty + \varepsilon_4, \frac{1}{L_1} - \varepsilon_2], \end{cases} \quad (\text{B.1})$$

where we omit the constants of integration since $\tilde{K}$ in equation (2.24) already serves the same purpose.

Furthermore, since $\mathcal{G}(\pi)$ is continuous on $[m + \varepsilon_1, \pi^\infty - \varepsilon_3]$ and $[\pi^\infty + \varepsilon_4, \frac{1}{L_1} - \varepsilon_2]$ and differentiable on the corresponding open intervals for arbitrarily small ε_3 and ε_4 , it is obvious that it is also well-defined on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$ ((a)), and satisfies $\mathcal{G} \in C^1$ on this domain ((b)).

(c) Given the discussion at the beginning of the proof, as well as the fact that $\frac{dA}{dx}(x) - \rho \frac{\sigma}{\sigma} \frac{dB}{dx}(x)$ is always continuous and positive on $x \in [m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2]$ by equation (A.3), it is obvious that due to the extreme value theorem, except the $P(x)$ term, all the other terms in the integrand together will stay positive and bounded on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2]$. Suppose $q \leq \frac{x + \frac{\gamma}{\kappa} [x - \frac{(\eta-A(x)) - \rho \frac{\sigma}{\sigma} (\tilde{\eta}-B(x))}{(1-\rho^2)\gamma\sigma^2}] + \frac{x}{(1-\rho^2)\gamma\sigma^2} [\frac{dA}{dx}(x) - \rho \frac{\sigma}{\sigma} \frac{dB}{dx}(x)]}{x[x - \frac{(\eta-A(x)) - \rho \frac{\sigma}{\sigma} (\tilde{\eta}-B(x))}{(1-\rho^2)\gamma\sigma^2}]} \leq Q$ for $Q \geq q > 0$ ²² and $x \in [m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2]$.

On the other hand, by the proof of Proposition 2.12 (a), we already show that under Assumption 2.11, we must have that $P'(\pi^\infty)$ is a strictly negative but bounded number. Meanwhile, by definition, we have that

$$P'(\pi^\infty) = \lim_{x \rightarrow \pi^\infty} \frac{P(x) - P(\pi^\infty)}{x - \pi^\infty} = \lim_{x \rightarrow \pi^\infty} \frac{P(x)}{x - \pi^\infty}.$$

Hence, since $\lim_{x \rightarrow \pi^\infty} \frac{P(x)}{x - \pi^\infty}$ converges to a negative number, we can conclude that the function $P(x)$ converges to 0 at the same rate as $\pi^\infty - x$ as x approaches π^∞ . Nevertheless, it is well-known that $\lim_{\pi \nearrow \pi^\infty} \int_a^\pi \frac{1}{\pi^\infty - x} dx = \infty$ for any $a < \pi^\infty$ and $\lim_{\pi \searrow \pi^\infty} \int_\pi^a \frac{1}{\pi^\infty - x} dx = -\infty$ for any $a > \pi^\infty$. Thus, we must have that (noticing that $P(x) > 0$ on $[m + \varepsilon_1, \pi^\infty)$ and $P(x) < 0$ on $(\pi^\infty, \frac{1}{L_1} - \varepsilon_2]$ by Corollary 2.13)

$$\lim_{\pi \nearrow \pi^\infty} \mathcal{G}(\pi) \geq \lim_{\pi \nearrow \pi^\infty} \int_{m+\varepsilon_1}^{\pi} \frac{q}{P(x)} dx = \infty, \quad \text{and that} \quad \lim_{\pi \searrow \pi^\infty} \mathcal{G}(\pi) \geq - \lim_{\pi \searrow \pi^\infty} \int_{\pi}^{\frac{1}{L_1} - \varepsilon_2} \frac{q}{P(x)} dx = \infty,$$

which prove the claim in (c).

²²It is easy to show the lower bound is strictly positive since the denominator is strictly positive and bounded from above while the numerator is greater than $m + \varepsilon_1$, which is strictly positive.

(d) Given the discussion in the proof of (c), we have already concluded that all the other terms in the integrand of $\mathcal{G}$ together is strictly positive and bounded. Therefore, it is clear that the sign of $P(\pi)$ dominates the sign of $\mathcal{G}'$. Thus the claim in (d) follows from Corollary 2.13.

(e) By (a), (b) and (d), we have shown that the function $\mathcal{G}$ is a well-defined continuously differentiable function on $[m + \varepsilon_1, \frac{1}{L_1} - \varepsilon_2] \setminus \{\pi^\infty\}$, which is strictly increasing on $[m + \varepsilon_1, \pi^\infty)$ and strictly decreasing on $(\pi^\infty, \frac{1}{L_1} - \varepsilon_2]$. The claim in (d) directly follows from the inverse function theorem. $\square$

Proof of Proposition 2.15. (a) By the inverse function theorem, the two inverses defined in Proposition 2.14 (e) are unique. Then by applying the correct inverse function to both sides of equation (2.25) in terms of $\pi(T)$'s value, we have the unique $\pi(t)$ as defined in (a), which is well-defined and continuously differentiable by Proposition 2.14 (e).

(b) Upper bound follows from the fact that when $\pi(T) < \pi^\infty$ we should take the inverse function $\mathcal{G}^{-1}$ of $\mathcal{G}$ defined on $[m + \varepsilon_1, \pi^\infty)$. Monotonicity follows from Proposition 2.14 (d) and the fact that the right-hand side of equation (2.25) is strictly decreasing with respect to t since $\overline{K} < 0$. Lower bound is directly implied by the upper bound and the monotonicity results.

(c) By symmetry with (b).

(d) Trivial by observing the ODE (2.19). $\square$

Proof of Proposition 2.16. (a) It can be shown that $\pi^{(\omega_1)}(T) > \pi^{(\omega_2)}(T)$. Notice that $\pi(t) \in C^1$ uniquely exists by Proposition 2.15 and satisfies the ODE (2.19). Then from $t = T$ backwards, the paths of $\pi^{(\omega_1)}(t)$ and $\pi^{(\omega_2)}(t)$ are both well-defined, and they either never intersect or intersect at some time point(s).

Suppose the two paths never intersect. Then given the fact that both paths are continuous, the ordering induced by their terminal conditions must be preserved, since otherwise the intermediate value theorem will imply that the two paths must intersect at least once. Therefore, we must have $\pi^{(\omega_1)}(t) > \pi^{(\omega_2)}(t)$.

Suppose the two paths do intersect at some points in $[0, T]$. Then denote the set of all the intersection time points by $\widehat{\mathcal{T}} := \{\hat{t} \in [0, T] | \pi^{(\omega_1)}(\hat{t}) = \pi^{(\omega_2)}(\hat{t})\}$ and denote its supremum by $\bar{\hat{t}} := \sup \widehat{\mathcal{T}}$. Due to the same arguments as the proof of Proposition 2.12 (b), we can show that $\pi^{(\omega_1)}(\bar{\hat{t}}) = \pi^{(\omega_2)}(\bar{\hat{t}})$ and $\bar{\hat{t}} < T$. For $t \in (\bar{\hat{t}}, T]$, since $\pi^{(\omega_1)}(t)$ and $\pi^{(\omega_2)}(t)$ do not intersect, $\pi^{(\omega_1)}(t) > \pi^{(\omega_2)}(t)$ must hold due to their continuity. On the other hand, for $t \in [0, \bar{\hat{t}}]$, since $\pi^{(\omega_1)}(t)$ and $\pi^{(\omega_2)}(t)$ have the same terminal condition on this sub-interval and follow the same ODE, their paths must be identical due to the uniqueness of $\pi(t)$ by Proposition 2.15, i.e., $\pi^{(\omega_1)}(t) = \pi^{(\omega_2)}(t)$ must hold.

Therefore, we can conclude that $\pi^{(\omega_1)}(t) \geq \pi^{(\omega_2)}(t)$.

(b) By Proposition 2.4, we already know that $\frac{d\tilde{\pi}}{d\pi} < 0$ on $\pi \in \mathbb{R}$. Therefore, for each $t \in [0, T]$ fixed, since $\pi^{(\omega_1)}(t) \geq \pi^{(\omega_2)}(t)$ by (a), we must have $\tilde{\pi}^{(\omega_1)}(t) \leq \tilde{\pi}^{(\omega_2)}(t)$;

(c) Observing equation (2.18), we can see that for each $t \in [0, T]$ fixed, $(\frac{c}{x})(t)$ is strictly increasing in $\pi(t)$ on due to Proposition 2.5 and equation (A.3). Therefore, for all $t \in [0, T]$ we must also have $(\frac{c}{x})^{(\omega_1)}(t) \geq (\frac{c}{x})^{(\omega_2)}(t)$. $\square$

Proof of Theorem 2.18. (a) follows from Proposition 2.15. (b) follows from Proposition 2.5. Given (a) and (b), (c) is then satisfied if the boundedness conditions (d) and (e) of Definition 2.17 are satisfied. Due to Proposition 2.5, π^* is bounded by $m > 0$ from below, which also implies that $c^* > 0$ by (2.18). Furthermore, Proposition 2.15 shows that $\pi^*(t)$ is also bounded from above (either by π^∞ or by $\pi(T)$, both of which are strictly lower than $\frac{1}{L_2}$), which in turn implies the boundedness of c^* and $\tilde{\pi}^*$ by (2.18) and Proposition 2.4. Therefore, $(c^*, \pi^*, \tilde{\pi}^*)$ is admissible. $\square$

C Numerical Solution Algorithm

Our numerical procedure directly implements the analytical characterization derived in Section 2. One of the main advantages of our theoretical results is that they reduce a highly nonlinear multidimensional system of equations to a sequence of *scalar* equations and a *one-dimensional* terminal-value problem. The solution can thus be obtained accurately and efficiently at low computational cost.

The procedure is as follows: For a given value of the first stock's portfolio share π , we use the implicit first-order condition (2.13) to determine $\tilde{\pi}(\pi)$. The resulting nonlinear scalar equation is solved using the MINPACK hybrid algorithm initialized at a sequence of starting values. To improve the numerical stability of this step, a candidate solution is accepted only if the absolute residual of the first-order condition is below a prescribed tolerance and the solution remains in the admissible domain of the model. The same admissibility restrictions are imposed when evaluating the reduced functions whenever $\tilde{\pi}(\pi)$ is computed.

Substitution of this implicit mapping yields the reduced functions $A(\pi)$, $B(\pi)$, $D(\pi)$, and $P(\pi)$. The first stock's optimal demand without asset tastes m and its infinite-horizon optimal demand π^∞ are then obtained from their respective scalar equations using Brent's method. This method is particularly efficient in our case because our theoretical analysis provides economically meaningful bounds for both quantities, allowing the relevant roots to be bracketed a priori and thereby avoiding costly unconstrained searches over the real line.

Finally, the finite-horizon optimal demand $\pi(t)$ is obtained by backward integration of the one-dimensional ODE characterized in equation (2.19), using an explicit Euler discretization with terminal condition $\pi(T)$. The second stock's optimal demand for both horizons is recovered from the implicit mapping $\tilde{\pi}(\pi^\infty)$ and $\tilde{\pi}(\pi(t))$, respectively.

The complete procedure is summarized in the following algorithm.

Algorithm 1.

Input: Model parameters, terminal condition $\pi(T)$, horizon T , and time step Δt .

1. Compute the constants κ , κ_α , $\tilde{K}$, and $\hat{K}$.
2. For any given π , solve the implicit first-order condition (2.13) using the MINPACK hybrid algorithm and obtain $\tilde{\pi}(\pi)$.
3. Using $\tilde{\pi}(\pi)$, evaluate $A(\pi)$, $B(\pi)$, $D(\pi)$, $P(\pi)$, and the analytical derivative required by the ODE (2.19).
4. Solve for m and π^∞ using Brent's method, and compute $\tilde{\pi}^\infty = \tilde{\pi}(\pi^\infty)$.
5. Construct the time grid $t_j = j\Delta t$, with $t_N = T$, and set $\pi(t_N) = \pi(T)$.
6. For $j = N, \dots, 1$, update the portfolio share backward according to

$$\pi(t_{j-1}) = \pi(t_j) - R(\pi(t_j))\Delta t,$$

where $R(\cdot)$ denotes the right-hand side of the ODE (2.19).

7. At each time-grid point, recover $\tilde{\pi}(t_j) = \tilde{\pi}(\pi(t_j))$.
8. Compute the total risky share in both horizons as $\Pi^{*,\infty} = \pi^\infty + \tilde{\pi}^\infty$ and $\Pi^*(t_j) = \pi(t_j) + \tilde{\pi}(t_j)$.

The numerical solution is checked for sensitivity to discretization and solver tolerances by repeating the computation with finer time grids and tighter tolerances.

D Proofs of the Verification Theorems

D.1 Proof of Theorem 2.19

We need to show that for all $(c, \pi, \tilde{\pi}) \in \mathcal{A}$, it holds that

$$\mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t^{c, \pi, \tilde{\pi}} \pi_t) dt + \omega e^{-\delta T} U(X_T^{c, \pi, \tilde{\pi}}, X_T \pi_T) \right] \leq G(0, X_0), \quad (\text{D.1})$$

and for our candidate $(c^*, \pi^*, \tilde{\pi}^*)$:

$$\mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t^*, X_t^* \pi_t^*) dt + \omega e^{-\delta T} U(X_T^*, X_T^* \pi_T^*) \right] = G(0, X_0). \quad (\text{D.2})$$

First, by Proposition 2.3, $(c^*, \pi^*, \tilde{\pi}^*)$ is the global maximizer of the HJB equation (G.3) on the region $(c, \pi, \tilde{\pi}) \in (0, \infty) \times (0, \frac{1}{L_1}) \times (-\infty, \frac{1}{L_2})$ and $\hat{L}_1 \pi + \hat{L}_2 \tilde{\pi} < 1$. Therefore, $(c^*, \pi^*, \tilde{\pi}^*)$ satisfy the HJB equation (without supremum) with equality, while other admissible strategies only satisfy the HJB equation (without supremum) with inequality.

Next, we consider the transformation $V(t, x) := e^{-\delta t} G(t, x)$, which satisfies

$$\begin{aligned} 0 = \sup_{c, \pi, \tilde{\pi}} \{ & V_t + x(r + \pi\eta + \tilde{\pi}\tilde{\eta})V_x - cV_x + 0.5x^2V_{xx}[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ & + \lambda[V(t, x(1 - L_1\pi)) - V(t, x)] + \tilde{\lambda}[V(t, x(1 - L_2\tilde{\pi})) - V(t, x)] \\ & + \hat{\lambda}[V(t, x(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})) - V(t, x)] + e^{-\delta t}u(c, x\pi)\}, \end{aligned} \quad (\text{D.3})$$

with a terminal condition $V(T, x) = e^{-\delta T} G(T, x) = \omega e^{-\delta T} U(x, x\pi(T))$.

Now, for every $(c, \pi, \tilde{\pi}) \in \mathcal{A}$ and its corresponding wealth process $X_t := X_t^{c, \pi, \tilde{\pi}}$, we apply Ito's formula to $V(t, X_t)$, which yields

$$\begin{aligned} dV = & \{V_t + V_x[X(r + \pi\eta + \tilde{\pi}\tilde{\eta}) - c] + 0.5V_{xx}X^2[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ & + \lambda[V(t, X(1 - L_1\pi)) - V(t, X)] + \tilde{\lambda}[V(t, X(1 - L_2\tilde{\pi})) - V(t, X)] \\ & + \hat{\lambda}[V(t, X(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})) - V(t, X)]\}dt \\ & + V_xX[(\sigma\pi + \rho\tilde{\sigma}\tilde{\pi})dW + \sqrt{1 - \rho^2}\tilde{\sigma}\tilde{\pi}d\widehat{W}] \\ & + [V(t, X(1 - L_1\pi)) - V(t, X)]dM + [V(t, X(1 - L_2\tilde{\pi})) - V(t, X)]d\widetilde{M} \\ & + [V(t, X(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})) - V(t, X)]d\widehat{M}, \end{aligned} \quad (\text{D.4})$$

where $M_t := N_t - \lambda t$, $\widetilde{M} := \tilde{N} - \tilde{\lambda}t$ and $\widehat{M} := \hat{\lambda}t$ are the compensated point processes, which are martingales.

Combining (D.4) with (D.3), we obtain

$$\begin{aligned} dV \leq & -e^{-\delta t}u(c, X\pi)dt + V_xX[(\sigma\pi + \rho\tilde{\sigma}\tilde{\pi})dW + \sqrt{1 - \rho^2}\tilde{\sigma}\tilde{\pi}d\widehat{W}] \\ & + [V(t, X(1 - L_1\pi)) - V(t, X)]dM + [V(t, X(1 - L_2\tilde{\pi})) - V(t, X)]d\widetilde{M} \\ & + [V(t, X(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})) - V(t, X)]d\widehat{M}. \end{aligned} \quad (\text{D.5})$$

Integrating (D.5) over $[0, T]$ yields

$$\begin{aligned} V(T, X_T) + \int_0^T e^{-\delta t}u(c_t, X_t\pi_t)dt \leq & V(0, X_0) + \int_0^T V_x(t, X_t)X_t(\sigma\pi_t + \rho\tilde{\sigma}\tilde{\pi}_t)dW_t \\ & + \int_0^T V_x(t, X_t)X_t\sqrt{1 - \rho^2}\tilde{\sigma}\tilde{\pi}_td\widehat{W}_t \\ & + \int_0^T [V(t, X_t(1 - L_1\pi_t)) - V(t, X_t)]dM_t \\ & + \int_0^T [V(t, X_t(1 - L_2\tilde{\pi}_t)) - V(t, X_t)]d\widetilde{M}_t \\ & + \int_0^T [V(t, X_t(1 - \hat{L}_1\pi_t - \hat{L}_2\tilde{\pi}_t)) - V(t, X_t)]d\widehat{M}_t, \end{aligned} \quad (\text{D.6})$$

where the equality holds for $(c^*, \pi^*, \tilde{\pi}^*)$.

The five integrands of the stochastic integrals are equal to

$$\left\{ \begin{array}{ll} V_x(t, X_t)X_t(\sigma\pi_t + \rho\tilde{\sigma}\tilde{\pi}_t) & = e^{-\delta t}X_t^{1-\gamma}f(t)^\kappa(\sigma\pi_t + \rho\tilde{\sigma}\tilde{\pi}_t) \\ V_x(t, X_t)X_t\sqrt{1-\rho^2}\tilde{\sigma}\tilde{\pi}_t & = e^{-\delta t}X_t^{1-\gamma}f(t)^\kappa\sqrt{1-\rho^2}\tilde{\sigma}\tilde{\pi}_t \\ [V(t, X_t(1-L_1\pi_t)) - V(t, X_t)] & = \frac{e^{-\delta t}}{1-\gamma}X_t^{1-\gamma}f(t)^\kappa[(1-L_1\pi_t)^{1-\gamma} - 1] \\ [V(t, X_t(1-L_2\tilde{\pi}_t)) - V(t, X_t)] & = \frac{e^{-\delta t}}{1-\gamma}X_t^{1-\gamma}f(t)^\kappa[(1-L_2\tilde{\pi}_t)^{1-\gamma} - 1] \\ [V(t, X_t(1-\hat{L}_1\pi_t - \hat{L}_2\tilde{\pi}_t)) - V(t, X_t)] & = \frac{e^{-\delta t}}{1-\gamma}X_t^{1-\gamma}f(t)^\kappa[(1-\hat{L}_1\pi_t - \hat{L}_2\tilde{\pi}_t)^{1-\gamma} - 1] \end{array} \right. \quad (D.7)$$

Note that by Definition 2.17, π , $\tilde{\pi}$, $(1-L_1\pi)^{1-\gamma}$, $(1-L_2\tilde{\pi})^{1-\gamma}$, $(1-\hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{1-\gamma}$ and the coefficients in the wealth dynamic (2.1) are all bounded. Besides, in Theorem 2.18 we have also concluded that f is smooth. Therefore, the five integrands in (D.7) are $\mathcal{L}^2$ -processes, which allows us to take expectations on both sides of (D.6):

$$\mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t^{c, \pi, \tilde{\pi}} \pi_t) dt \right] + \omega \mathbb{E} \left[e^{-\delta T} U(X_T^{c, \pi, \tilde{\pi}}, X_T^{c, \pi, \tilde{\pi}} \pi_T) \right] \leq V(0, X_0) = G(0, X_0),$$

which proves (D.1). Finally, (D.2) directly follows from the fact that the equality of the HJB equation holds for $(c^*, \pi^*, \tilde{\pi}^*)$. $\square$

D.2 Proof of Theorem 2.20

We need to show that for all $(c, \pi, \tilde{\pi}) \in \mathcal{A}$, it holds that

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} u(c_t, X_t^{c, \pi, \tilde{\pi}} \pi_t) dt \right] \leq G(X_0), \quad (D.8)$$

and for our candidate $(c^\infty, \pi^\infty, \tilde{\pi}^\infty)$:

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} u(c^\infty, X_t^* \pi^\infty) dt \right] = G(X_0). \quad (D.9)$$

We first prove (D.8).

Similar to the finite-horizon case, we can still define $V(t, x) := e^{-\delta t}G(x)$ (note that $G(x)$ itself is independent of time), and prove the same claim since the five integrands are still $\mathcal{L}^2$ -processes for any admissible strategies. Specifically, we have for all $(c, \pi, \tilde{\pi}) \in \mathcal{A}$

$$\mathbb{E} [V(T, X_T^{c, \pi, \tilde{\pi}})] + \mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t^{c, \pi, \tilde{\pi}} \pi_t) dt \right] \leq V(0, X_0),$$

or, equivalently

$$e^{-\delta T} \mathbb{E} \left[G(X_T^{c,\pi,\tilde{\pi}}) \right] + \mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t^{c,\pi,\tilde{\pi}} \pi_t) dt \right] \leq G(X_0). \quad (\text{D.10})$$

Note that this inequality (D.10) holds for all the admissible strategies. Therefore, it also holds that

$$\sup_{c,\pi,\tilde{\pi}} \left\{ e^{-\delta T} \mathbb{E} \left[G(X_T^{c,\pi,\tilde{\pi}}) \right] \right\} + \mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t^{c,\pi,\tilde{\pi}} \pi_t) dt \right] \leq G(X_0). \quad (\text{D.11})$$

Note further that the utility functional is always nonpositive. Hence, by multiplying -1 to the integrand in (D.11), we can apply the monotone convergence theorem to the second term on the left-hand side and let $T \rightarrow \infty$:

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t, X_t^{c,\pi,\tilde{\pi}} \pi_t) dt \right] = \mathbb{E} \left[\int_0^\infty e^{-\delta t} u(c_t, X_t^{c,\pi,\tilde{\pi}} \pi_t) dt \right].$$

Thus, (D.8) will hold if the following equation can hold:

$$\lim_{T \rightarrow \infty} \sup_{c,\pi,\tilde{\pi}} \left\{ e^{-\delta T} \mathbb{E} \left[G(X_T^{c,\pi,\tilde{\pi}}) \right] \right\} = 0. \quad (\text{D.12})$$

Since $G(x)$ is nonpositive, we have

$$\begin{aligned} 0 &\geq \sup_{c,\pi,\tilde{\pi}} \left\{ e^{-\delta T} \mathbb{E} \left[G(X_T^{c,\pi,\tilde{\pi}}) \right] \right\} = \sup_{c,\pi,\tilde{\pi}} \left\{ \mathbb{E} \left[e^{-\delta T} \frac{1}{1-\gamma} (X_T^{c,\pi,\tilde{\pi}})^{1-\gamma} \right] \right\} f^\kappa \\ &\geq \sup_{\pi,\tilde{\pi}} \left\{ \mathbb{E} \left[e^{-\delta T} \frac{1}{1-\gamma} (X_T^{0,\pi,\tilde{\pi}})^{1-\gamma} \right] \right\} f^\kappa. \end{aligned} \quad (\text{D.13})$$

Notice that the supremum in the last term of (D.13) is exactly the value function of a Merton's problem with bequest motive only. It is the easy to check that (let $e^{-\delta t} J(t, x) := \sup_{\pi,\tilde{\pi}} \{ \mathbb{E}_t [e^{-\delta T} \frac{1}{1-\gamma} (X_T^{\pi,\tilde{\pi}})^{1-\gamma}] \}$) be the indirect utility subject to the dynamic budget $dX = X[(r + \eta\pi + \tilde{\eta}\tilde{\pi})dt + (\sigma\pi + \rho\tilde{\sigma}\tilde{\pi})dW + \sqrt{1 - \rho^2\tilde{\sigma}\tilde{\pi}}d\tilde{W}] - X_-[L_1\pi_-dN + L_2\tilde{\pi}_-d\tilde{N} + (\hat{L}_1\pi_- + \hat{L}_2\tilde{\pi}_-)d\tilde{N}]$:

$$J(t, x) = \frac{1}{1-\gamma} x^{1-\gamma} e^{-(\gamma-1)\tilde{r}(T-t)},$$

where $\tilde{r} := r + \frac{\delta}{\gamma-1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})m - \frac{1}{2}(1 - \rho^2)\gamma\sigma^2m^2 - \frac{D(m)}{\gamma-1} - \frac{B(m)^2}{2\gamma\tilde{\sigma}^2} = P(m)$, with m being the solution of π to the Merton's problem with jumps but without preferences for stock (see Appendix F) and P being defined as in Proposition 2.6. By Assumption 2.7, $\tilde{r} > 0$, thus we obtain that as $T \rightarrow \infty$

$$\begin{aligned} 0 &\geq \sup_{c,\pi,\tilde{\pi}} \left\{ e^{-\delta T} \mathbb{E} \left[G(X_T^{c,\pi,\tilde{\pi}}) \right] \right\} \geq \sup_{\pi,\tilde{\pi}} \left\{ \mathbb{E} \left[e^{-\delta T} \frac{1}{1-\gamma} (X_T^{0,\pi,\tilde{\pi}})^{1-\gamma} \right] \right\} f^\kappa \\ &= \frac{1}{1-\gamma} X_0^{1-\gamma} e^{-(\gamma-1)\tilde{r}(T-t)} f^\kappa \rightarrow 0. \end{aligned}$$

Therefore, equation (D.12) holds, which proves (D.8). Next, we proceed to prove (D.9). Note that (D.10) holds with equality for the global maximizer $(c^\infty, \pi^\infty, \tilde{\pi}^\infty)$. Hence, it suffices to show the following:

$$\lim_{T \rightarrow \infty} e^{-\delta T} \mathbb{E}[G(X_T^*)] = \frac{1}{1-\gamma} f^\kappa \lim_{T \rightarrow \infty} e^{-\delta T} \mathbb{E}[(X_T^*)^{1-\gamma}] = 0.$$

Recall the dynamic budget (G.1) evaluated at $(c^\infty, \pi^\infty, \tilde{\pi}^\infty)$:

$$\begin{aligned} dX^* = & X^*[(r + \eta\pi^\infty + \tilde{\eta}\tilde{\pi}^\infty - \frac{c^\infty}{X^*})dt + (\sigma\pi^\infty + \rho\tilde{\sigma}\tilde{\pi}^\infty)dW + \sqrt{1-\rho^2}\tilde{\sigma}\tilde{\pi}^\infty d\widehat{W}] \\ & - X_-[L_1\pi^\infty dN + L_2\tilde{\pi}^\infty d\tilde{N} + (\widehat{L}_1\widehat{\pi}^\infty + \widehat{L}_2\tilde{\pi}^\infty)d\widehat{N}]. \end{aligned}$$

Then, by applying Ito's formula to $(X^*)^{1-\gamma}$, we obtain

$$\begin{aligned} d(X^*)^{1-\gamma} = & (1-\gamma)(X^*)^{1-\gamma}[(r + \eta\pi^\infty + \tilde{\eta}\tilde{\pi}^\infty - \frac{c^\infty}{X^*} - \frac{\gamma}{2}(\sigma\pi^\infty + \rho\tilde{\sigma}\tilde{\pi}^\infty)^2 - \frac{\gamma}{2}(1-\rho^2)\tilde{\sigma}^2\tilde{\pi}^{\infty 2})dt \\ & + (\sigma\pi^\infty + \rho\tilde{\sigma}\tilde{\pi}^\infty)dW + \sqrt{1-\rho^2}\tilde{\sigma}\tilde{\pi}^\infty d\widehat{W}] \\ & + (X^*)^{1-\gamma}\{[(1-L_1\pi^\infty)^{1-\gamma} - 1]dN + [(1-L_2\tilde{\pi}^\infty)^{1-\gamma} - 1]d\tilde{N} \\ & + [(1-\widehat{L}_1\pi^\infty - \widehat{L}_2\tilde{\pi}^\infty)^{1-\gamma} - 1]d\widehat{N}\}. \end{aligned}$$

Given that the Brownian motions and the Poisson processes must be independent, we can compute

$$e^{-\delta T} \mathbb{E}[(X_T^*)^{1-\gamma}] = X_0^{1-\gamma} e^{\phi T},$$

where ϕ is a constant defined as

$$\begin{aligned} \phi = & (1-\gamma)(r + \eta\pi^\infty + \tilde{\eta}\tilde{\pi}^\infty - \frac{c^\infty}{X^*} - \frac{\gamma}{2}(\sigma\pi^\infty + \rho\tilde{\sigma}\tilde{\pi}^\infty)^2 - \frac{\gamma}{2}(1-\rho^2)\tilde{\sigma}^2\tilde{\pi}^{\infty 2}) - \delta \\ & + \lambda[(1-L_1\pi^\infty)^{1-\gamma} - 1] + \tilde{\lambda}[(1-L_2\tilde{\pi}^\infty)^{1-\gamma} - 1] + \widehat{\lambda}[(1-\widehat{L}_1\pi^\infty - \widehat{L}_2\tilde{\pi}^\infty)^{1-\gamma} - 1] \\ = & (1-\gamma)(r + \frac{\delta}{\gamma-1} + \eta\pi^\infty + \tilde{\eta}\tilde{\pi}^\infty - \frac{c^\infty}{X^*} - \frac{\gamma}{2}(\sigma\pi^\infty + \rho\tilde{\sigma}\tilde{\pi}^\infty)^2 - \frac{\gamma}{2}(1-\rho^2)\tilde{\sigma}^2\tilde{\pi}^{\infty 2} - \frac{D(\pi^\infty)}{\gamma-1}). \end{aligned} \tag{D.14}$$

Substituting (2.26) and (2.27) into (D.14), we then have

$$\phi = \underbrace{(1-\gamma)}_{<0} \underbrace{\{P(\pi^\infty) - (\tilde{K} + \frac{\alpha}{1-\alpha})((1-\rho^2)\gamma\sigma^2\pi^\infty - [(\eta - A(\pi^\infty)) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\pi^\infty))])\pi^\infty\}}_{\substack{= \frac{1}{(1-\alpha)(1-\gamma)} < 0}} > 0 < 0.$$

where the last term is strictly positive due to Proposition 2.5. Hence, we can finally conclude that

$$\lim_{T \rightarrow \infty} e^{-\delta T} \mathbb{E}[G(X_T^*)] = \frac{1}{1-\gamma} X_0^{1-\gamma} f^\kappa \lim_{T \rightarrow \infty} e^{\phi T} = 0,$$

which proves (D.9). $\square$

E Calibration Details

The average annual risk-free rate is estimated to be 0.30%. To identify jump events, we follow an approach similar to Liu et al. (2003). The full-sample annual volatility of the MSCI KLD 400 Social Index is 19.17%. We classify daily returns below -4.80% as jumps, corresponding approximately to returns more than four daily standard deviations below the near-zero mean under normality. This yields 21 jumps. Similarly, the full-sample annual volatility of the S&P 500 Energy Index is 26.86%, implying a jump threshold of -6.80% and yielding 20 jumps. A jump occurring in both indices on the same trading day is classified as a co-jump; all other jumps are classified as idiosyncratic. We identify 13 co-jumps, implying a co-jump intensity of 0.4274 (13/30.42). The idiosyncratic jump intensities are 0.2630 (8/30.42) and 0.2301 (7/30.42) for the MSCI KLD 400 Social Index and the S&P 500 Energy Index, respectively. The corresponding average co-jump and idiosyncratic jump sizes are -7.44% and -5.74% for the MSCI KLD 400 Social Index and -11.80% and -7.43% for the S&P 500 Energy Index. Finally, we estimate the diffusive parameters after excluding all jump days. For the MSCI KLD 400 Social Index, the average annual return parameter is 14.89%, the annual excess return parameter is 15.06%, and the annual volatility is 18.22%. For the S&P 500 Energy Index, the corresponding values are 17.82%, 17.99%, and 25.31%. The correlation between the two indices' diffusive returns is 0.5769. For the preference parameters, the classical asset pricing literature (e.g., Campbell and Cochrane, 1999; Barro, 2006, and the references therein) typically assumes a coefficient of relative risk aversion between 2 and 5 and a time preference rate between 0.03 and 0.12. In our baseline calibration, we therefore set $\gamma = 5$ and $\delta = 0.03$. We further choose $\alpha = 0.90$ and $\tilde{\alpha} = 1.00$, implying a moderate preference for green stocks in instantaneous utility and no such preference in the bequest motive. The results are similar for other choices of $\tilde{\alpha}$ (e.g., $\tilde{\alpha} = 0.99$). In addition, we set $\pi(T) = 52.00\%$ in the baseline calibration implying a slightly decreasing total risky share in the finite-horizon case.²³ The time horizon is 50 years. Table 1 summarizes the baseline calibration.²⁴

F Model Without Preferences for Stock

Consider a two-stock Merton (1969, 1971)'s consumption-portfolio choice problem, where the stocks are exposed to jump risks:

$$\begin{aligned} dM &= Mrdt, \\ dS &= S[\mu dt + \sigma dW] - S_-[L_1 dN + \hat{L}_1 d\hat{N}], \end{aligned}$$

²³Notice that by (2.20), there is a one-to-one correspondence between ω and $\pi(T)$. Therefore, it is both sufficient and more intuitive to specify the value of $\pi(T)$ directly rather than work with ω .

²⁴Notice that the excess return parameter does not include the compensators of the jump processes, which explains its magnitude.

$$d\tilde{S} = \tilde{S}[\tilde{\mu}dt + \tilde{\sigma}(\rho dW + \sqrt{1 - \rho^2}d\widehat{W})] - \tilde{S}_-[L_2d\tilde{N} + \widehat{L}_2d\widehat{N}],$$

where all the parameters and stochastic processes are defined the same as those presented in Section 2.1.

The agent maximizes

$$\sup_{c, \pi, \tilde{\pi}} \mathbb{E} \left[\int_0^T e^{-\delta t} u(c_t) dt + \tilde{\omega} e^{-\delta T} U(X_T) \right],$$

subject to the dynamic budget constraint

$$\begin{aligned} dX = & X[(r + \eta\pi + \tilde{\eta}\tilde{\pi})dt + (\sigma\pi + \rho\tilde{\sigma}\tilde{\pi})dW + \sqrt{1 - \rho^2}\tilde{\sigma}\tilde{\pi}d\widehat{W}] \\ & - X_-[L_1\pi_-dN + L_2\tilde{\pi}_-d\tilde{N} + (\widehat{L}_1\pi_- + \widehat{L}_2\tilde{\pi}_-)d\widehat{N}] - cdt, \end{aligned}$$

where the agent's utility functions are given by

$$u(c) = \frac{1}{1 - \gamma} c^{1 - \gamma}, \quad U(x) = \frac{1}{1 - \gamma} x^{1 - \gamma}.$$

Then, define the indirect utility function as

$$G(t, x) := \sup_{c, \pi, \tilde{\pi}} \left\{ e^{\delta t} \left[\int_t^T e^{-\delta s} \mathbb{E}_t[u(c_s)] ds + \tilde{\omega} e^{-\delta T} \mathbb{E}_t[U(X_T)] \right] \right\},$$

and the terminal condition is $G(T, x) = \tilde{\omega}U(x)$.

The HJB equation of the problem reads:

$$\begin{aligned} 0 = & \sup_{c, \pi, \tilde{\pi}} \{ G_t - \delta G + x(r + \pi\eta + \tilde{\pi}\tilde{\eta})G_x - cG_x + 0.5x^2G_{xx}[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ & + \lambda[G(t, x(1 - L_1\pi)) - G(t, x)] + \tilde{\lambda}[G(t, x(1 - L_2\tilde{\pi})) - G(t, x)] \\ & + \widehat{\lambda}[G(t, x(1 - \widehat{L}_1\pi - \widehat{L}_2\tilde{\pi})) - G(t, x)] + u(c) \}. \end{aligned}$$

To solve the problem, we first consider the conjecture

$$G(t, x) = \frac{1}{1 - \gamma} x^{1 - \gamma} f(t)^k \tag{F.1}$$

with k being a constant to choose later and the terminal condition being $f(T) = \tilde{\omega}^{\frac{1}{k}}$. Then given the conjecture (F.1), we can simplify the additional terms that have occurred because of jumps:

$$\begin{aligned} 0 = & \sup_{c, \pi, \tilde{\pi}} \{ G_t - \delta G + x(r + \pi\eta + \tilde{\pi}\tilde{\eta})G_x - cG_x + 0.5x^2G_{xx}[(\pi\sigma + \rho\tilde{\pi}\tilde{\sigma})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] \\ & + \overline{D}(\pi, \tilde{\pi})G + u(c) \}, \end{aligned}$$

where we define

$$\overline{D}(\pi, \tilde{\pi}) := \lambda[(1 - L_1\pi)^{1-\gamma} - 1] + \tilde{\lambda}[(1 - L_2\tilde{\pi})^{1-\gamma} - 1] + \hat{\lambda}[(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{1-\gamma} - 1].$$

The first-order condition with respect to c is

$$c = G_x^{-\frac{1}{\gamma}},$$

and the first-order conditions with respect to π and $\tilde{\pi}$ are:

$$\begin{cases} \pi &= \frac{1}{1-\rho^2} \left[\frac{(1-\gamma)\overline{A}(\pi, \tilde{\pi})G - \eta x G_x}{\sigma^2 x^2 G_{xx}} - \rho \frac{(1-\gamma)\overline{B}(\pi, \tilde{\pi})G - \tilde{\eta} x G_x}{\sigma \tilde{\sigma} x^2 G_{xx}} \right] \\ \tilde{\pi} &= \frac{1}{1-\rho^2} \left[\frac{(1-\gamma)\overline{B}(\pi, \tilde{\pi})G - \tilde{\eta} x G_x}{\tilde{\sigma}^2 x^2 G_{xx}} - \rho \frac{(1-\gamma)\overline{A}(\pi, \tilde{\pi})G - \eta x G_x}{\sigma \tilde{\sigma} x^2 G_{xx}} \right] \end{cases},$$

where we define

$$\begin{cases} \overline{A}(\pi, \tilde{\pi}) &= \lambda L_1(1 - L_1\pi)^{-\gamma} + \hat{\lambda} \hat{L}_1(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma} \\ \overline{B}(\pi, \tilde{\pi}) &= \tilde{\lambda} L_2(1 - L_2\tilde{\pi})^{-\gamma} + \hat{\lambda} \hat{L}_2(1 - \hat{L}_1\pi - \hat{L}_2\tilde{\pi})^{-\gamma} \end{cases},$$

Finally, by substituting the conjecture (F.1) into the first-order conditions and the HJB equation (choose $k = \gamma$), we obtain the following nonlinear equation system:

The optimal consumption-wealth ratio:

$$\frac{c}{x} = f^{-1}.$$

The optimal demand for the first stock:²⁵

$$\pi = \frac{1}{(1 - \rho^2)\gamma\sigma^2} [(\eta - \overline{A}) - \rho \frac{\sigma}{\tilde{\sigma}} (\tilde{\eta} - \overline{B})]. \quad (\text{F.2})$$

The optimal demand for the second stock:

$$\tilde{\pi} = \frac{1}{(1 - \rho^2)\gamma\tilde{\sigma}^2} [(\tilde{\eta} - \overline{B}) - \rho \frac{\tilde{\sigma}}{\sigma} (\eta - \overline{A})]. \quad (\text{F.3})$$

The reduced HJB equation:

$$\begin{aligned} f_t - \frac{\gamma-1}{\gamma} \left\{ \left[r + \frac{\delta}{\gamma-1} + \frac{1}{2(1-\rho^2)\gamma\sigma^2\tilde{\sigma}^2} \{ [(\eta - \overline{A})\tilde{\sigma} - (\tilde{\eta} - \overline{B})\sigma][(\eta + \overline{A})\tilde{\sigma} - (\tilde{\eta} + \overline{B})\sigma] \right. \right. \\ \left. \left. + 2(1-\rho)\sigma\tilde{\sigma}(\eta\tilde{\eta} - \overline{AB}) \right\} - \frac{\overline{D}}{\gamma-1} \right\} f + 1 = 0, \end{aligned} \quad (\text{F.4})$$

with the terminal condition $f(T) = \tilde{\omega}^{\frac{1}{\gamma}}$.

²⁵Notice that if $\lambda = \tilde{\lambda} = \hat{\lambda} = 0$, i.e., in a model without jumps, we immediately recover the classical solution of the first stock's optimal demand in Merton (1969, 1971)'s problem without jumps or preferences for stock: $\tilde{m} = \frac{\eta - \rho \frac{\sigma}{\tilde{\sigma}} \tilde{\eta}}{(1-\rho^2)\gamma\sigma^2}$.

An important remark on the above results is that observing equations (F.2) and (F.3), we can derive an alternative representation of $\tilde{\pi}$ as follows:

$$\tilde{\pi} = \frac{\tilde{\eta} - \bar{B}}{\gamma\tilde{\sigma}^2} - \rho\frac{\sigma}{\tilde{\sigma}}\pi. \quad (\text{F.5})$$

We stress the importance of (F.5) because this functional form of $\tilde{\pi}$ carries over to our baseline model in Section 2, which helps us build the link between the solutions to our model and to the standard Merton's problem without preference for stock. In particular, it is worth noting that Proposition 2.4 also applies to this classic case under Assumption 2.1 (b).

On the other hand, notice that the two equations (F.2) and (F.3) for the optimal stock demands constitute a nonlinear system with two equations and two unknowns. The existence and uniqueness of the solution to this equation system is easy to establish (again, under Assumption 2.1 (b)):

By substituting the implicit function $\tilde{\pi}$ in Proposition 2.4 into (F.2) and replace $\bar{A}$ and $\bar{B}$ with A and B as in Section 2.2, we can reduce this system to a single equation. Then, for any $\underline{M} \in (-\infty, \frac{1}{L_1})$ fixed, we can show that the right-hand side of (F.2) is finite, i.e., $\frac{(\eta - A(\underline{M})) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\underline{M}))}{(1-\rho^2)\gamma\sigma^2} > -\infty$, because by (F.5), for any $\pi \in (-\infty, \frac{1}{L_1})$ fixed, we have $|B(\pi)| = |\tilde{\eta} - \gamma\tilde{\sigma}^2\tilde{\pi}(\pi) - \gamma\rho\sigma\tilde{\sigma}\pi| \leq \tilde{\eta} + \gamma\tilde{\sigma}^2|\tilde{\pi}(\pi)| + \gamma\rho\sigma\tilde{\sigma}|\pi| < \infty$, and thus $|A(\pi)| < \infty$ also holds by (F.3), where $|\tilde{\pi}(\pi)| < \infty$ due to Proposition 2.4. Next, we define

$$\underline{M}' := \begin{cases} \underline{M}, & \text{if } \frac{(\eta - A(\underline{M})) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\underline{M}))}{(1-\rho^2)\gamma\sigma^2} \geq \underline{M} \\ \frac{(\eta - A(\underline{M})) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\underline{M}))}{(1-\rho^2)\gamma\sigma^2}, & \text{if } \frac{(\eta - A(\underline{M})) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\underline{M}))}{(1-\rho^2)\gamma\sigma^2} < \underline{M} \end{cases}.$$

By construction, we must have that $\frac{(\eta - A(\underline{M}')) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\underline{M}'))}{(1-\rho^2)\gamma\sigma^2} \geq \underline{M}'$, and that as the left-hand side of (F.2) increases from $\underline{M}'$ to $\frac{1}{L_1}$, the right-hand side (after substituting in $\tilde{\pi}(\pi)$) monotonically decreases from $\frac{(\eta - A(\underline{M}')) - \rho\frac{\sigma}{\tilde{\sigma}}(\tilde{\eta} - B(\underline{M}'))}{(1-\rho^2)\gamma\sigma^2}$ to $-\infty$ (due to the fact that if $\pi \nearrow \frac{1}{L_1}$, only the function $A(\pi)$ will explode; see the proof of Lemma 2.10 for a detailed discussion), where we also use the strict monotonicity result of $A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi)$, which we will prove later by (A.3).

In addition, both sides of (F.2) are continuous by Proposition 2.4. Hence, by the intermediate value theorem, there must be one unique $\pi \in [\underline{M}', \frac{1}{L_1})$ that solves this equation, which further implies a unique $\tilde{\pi}$ by Proposition 2.4.

Therefore, we can always solve (F.2) and (F.3) numerically to obtain two constant investment demands. Although the above HJB equation (F.4) seems to be complicated, all its coefficients can be reduced to constants and solved in a straightforward way as long as we have obtained the values of π and $\tilde{\pi}$.

G Model without Jumps

This section discusses a simplified case, i.e., a model without jumps in stock prices. As we will see in the following, in this case all the previous results will become explicit, which helps to illustrate our main results in a clearer and more intuitive way.

Specifically, consider the same consumption-portfolio choice problem as the one presented in Section 2, but now the assets prices' dynamics are as follows:

$$\begin{aligned} dM &= Mrdt, \\ dS &= S[\mu dt + \sigma dW], \\ d\tilde{S} &= \tilde{S}[\tilde{\mu} dt + \tilde{\sigma}(\rho dW + \sqrt{1 - \rho^2} d\widehat{W})], \end{aligned}$$

where r , μ , $\tilde{\mu}$ and $\sigma > 0$, $\tilde{\sigma} > 0$ have the same definitions and are still assumed to be constants as in Section 2. W and $\widehat{W}$ are two independent Brownian motions, and ρ is the constant instantaneous correlation between them.

The agent's wealth X thus satisfies the dynamic budget constraint

$$dX = X[(r + \eta\pi + \tilde{\eta}\tilde{\pi})dt + (\sigma\pi + \rho\tilde{\sigma}\tilde{\pi})dW + \sqrt{1 - \rho^2}\tilde{\sigma}\tilde{\pi}d\widehat{W}] - cdt \quad (\text{G.1})$$

where $\eta := \mu - r$ and $\tilde{\eta} := \tilde{\mu} - r$. Besides, π and $\tilde{\pi}$ are the proportions of the wealth that are invested in the stocks, and c denotes consumption. In addition, we define $\tilde{m} := \frac{\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta}}{(1 - \rho^2)\gamma\sigma^2}$, which is the solution of the optimal demand for the first stock in the classical Merton (1969, 1971)'s problem without jumps or preferences for stock.

The agent's indirect utility reads

$$G(t, x) := \sup_{c, \pi, \tilde{\pi}} \left\{ e^{\delta t} \mathbb{E}_t \left[\int_t^T e^{-\delta s} u(c_s, X_s \pi_s) ds + \tilde{\omega} e^{-\delta T} U(X_T, X_T \pi_T) \right] \right\}, \quad (\text{G.2})$$

where the utility functions $u(c, \ell)$ and $U(c, \ell)$ are also defined in the same way as in Section 2, and $\tilde{\omega} > 0$ is the strength of the agent's bequest motive relative to her life-time (additive) utility.

The HJB equation for the maximization problem (G.2) with respect to the dynamic budget constraint (G.1) can be written as

$$0 = \sup_{c, \pi, \tilde{\pi}} \{ G_t - \delta G + x(r + \pi\eta + \tilde{\pi}\tilde{\eta})G_x - cG_x + 0.5x^2 G_{xx} [(\pi\sigma + \rho\tilde{\sigma}\tilde{\pi})^2 + (1 - \rho^2)\tilde{\pi}^2\tilde{\sigma}^2] + u(c, x\pi) \}. \quad (\text{G.3})$$

To find the solution to this problem, we use the same conjecture as (2.4) with the terminal condition

$$f(T) = \tilde{\omega}^{\frac{1}{k}} \pi(T)^{\frac{(1 - \tilde{\alpha})(1 - \gamma)}{k}}. \quad (\text{G.4})$$

By comparing equation (G.3) with (2.3), it is obvious that the original problem in Section 2 will reduce to the above problem if we set $\lambda = \tilde{\lambda} = \hat{\lambda} = 0$ (which implies $\overline{D}(\pi, \tilde{\pi}) = \overline{A}(\pi, \tilde{\pi}) = \overline{B}(\pi, \tilde{\pi}) = 0$ for any π and $\tilde{\pi}$). Therefore, in the following, we only summarize the major results for the current model without going into the details of the derivations again. As a matter of fact, all the derivations are exactly the same as those discussed in the previous section. In particular, in the following we keep assuming that Assumption 2.1 holds. In addition, we further assume the following condition holds:

Assumption G.1. *First stock's optimal demand in the case without jumps or preferences for stock, denoted by $\tilde{m}$, is strictly positive: $\tilde{m} > 0$.*

Before we proceed, we again temporarily assume that Assumption 2.2 holds for the f function and all the first-order conditions in the current model without jumps, which can be verified later when relevant verification theorems are proven.

Setting $k = \kappa := 1 + \alpha(\gamma - 1)$, the optimal consumption-wealth ratio is

$$\frac{c}{x} = \frac{\alpha}{1 - \alpha} [(1 - \rho^2)\gamma\sigma^2\pi - (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})]\pi,$$

whereas the optimal demand for the second stock is

$$\tilde{\pi} = \frac{\tilde{\eta}}{\gamma\tilde{\sigma}^2} - \rho\frac{\sigma}{\tilde{\sigma}}\pi, \quad (\text{G.5})$$

Note that equation (G.5) is exactly the counterpart of the implicit function $\tilde{\pi}(\pi)$ as defined in Proposition 2.4, which becomes linear in the no-jump model and its slope $-\rho\frac{\sigma}{\tilde{\sigma}}$ can also be viewed as a special case if we set all the jump intensities to zero in (2.14). The first-order condition with respect to π is

$$f = \frac{\frac{1-\alpha}{\alpha}\alpha^{\frac{1}{\kappa}}}{[(1 - \rho^2)\gamma\sigma^2\pi - (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})]\pi^{\frac{\gamma}{\kappa}}}. \quad (\text{G.6})$$

The reduced HJB equation reads

$$\frac{\kappa}{1 - \gamma}f_t + \{r + \frac{\delta}{\gamma - 1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})\pi - \frac{1}{2}(1 - \rho^2)\gamma\sigma^2\pi^2 + \tilde{K}[(1 - \rho^2)\gamma\sigma^2\pi - (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})]\pi\}f = 0.$$

The ODE characterization of the optimal demand for the first stock is given by the following proposition:²⁶

Proposition G.2. *(ODE characterization of the optimal demand for the first stock). The optimal demand for the first stock is given by the solution to the following ODE:*

$$\frac{d\pi}{dt} = -\frac{\gamma - 1}{\kappa} \frac{P_2(\pi)\pi(\pi - \tilde{m})}{\pi + \frac{\gamma}{\kappa}(\pi - \tilde{m})}, \quad (\text{G.7})$$

²⁶The proof is omitted since it follows the same steps as the proof of Proposition 2.6.

where $P_2(\pi)$ is a second-order polynomial with respect to π :

$$P_2(\pi) = r + \frac{\delta}{\gamma - 1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} + \frac{1}{2}(1 - \rho^2)\gamma\sigma^2\pi^2 + (\tilde{K} - 1)(1 - \rho^2)\gamma\sigma^2(\pi - \tilde{m})\pi,$$

and $\tilde{K} := \frac{\kappa}{(1-\gamma)(1-\alpha)} < 0$.

Analogously, we impose the following assumption to the P_2 function:

Assumption G.3. Assume that $P_2(\tilde{m}) > 0$, or equivalently, the interest rate is not too low, which satisfies

$$r > -\frac{\delta}{\gamma - 1} - \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} - \frac{1}{2}(1 - \rho^2)\gamma\sigma^2\tilde{m}^2.$$

Following the same procedure as in Section 2, it is easy to verify that the above strategy gives a global maximum of the HJB equation. Moreover, from (G.6) we can prove that $\pi(t) > \tilde{m}$ for all $t \in [0, T]$, which in turn implies that the slope of $\pi(t)$ must have opposite sign to the sign of $P_2(\pi(t))$ by equation (G.7).

Finally, the terminal condition of the ODE (G.7) is given by combining (G.6) with (G.4) and letting $t = T$:

$$f(T) = \tilde{\omega}^{\frac{1}{\kappa}} \pi(T)^{\frac{(1-\tilde{\alpha})(1-\gamma)}{\kappa}} = \frac{\frac{1-\alpha}{\alpha} \alpha^{\frac{1}{\kappa}}}{[(1 - \rho^2)\gamma\sigma^2\pi(T) - (\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta})]\pi(T)^{\frac{\gamma}{\kappa}}},$$

which can be simplified to

$$(1 - \rho^2)\gamma\sigma^2[\pi(T) - \tilde{m}] = \tilde{\omega}^{-\frac{1}{\kappa}} \frac{1 - \alpha}{\alpha} \alpha^{\frac{1}{\kappa}} \pi(T)^{-\frac{\kappa\tilde{\alpha}}{\kappa}}, \quad (\text{G.8})$$

where $\kappa_{\tilde{\alpha}} := 1 + \tilde{\alpha}(\gamma - 1) > 0$ by Assumption 2.1 (a). Note that $\pi(T) > \tilde{m}$ also holds by (G.8).

While most of the calculations and derivations in Section 2 directly carry over to this section, the simplified model as we present in the above also allows us to obtain more explicit results. In particular, notice that in this case, the P function reduces to a parabolic function P_2 with respect to π , which is well-defined and has well-known properties on the real line. We proceed by rewriting $P_2(\pi)$ as follows:

$$P_2(\pi) = \hat{K}(\pi^2 + p\pi + q)$$

with

$$\begin{aligned} \hat{K} &:= (\tilde{K} - \frac{1}{2})(1 - \rho^2)\gamma\sigma^2 < 0, \\ p &:= \frac{1}{\hat{K}}(1 - \tilde{K})(\eta - \rho\frac{\sigma}{\tilde{\sigma}}\tilde{\eta}) < 0, \end{aligned}$$

$$q := \frac{1}{\widehat{K}} \left(r + \frac{\delta}{\gamma - 1} + \frac{\tilde{\eta}^2}{2\gamma\tilde{\sigma}^2} \right) < 0.$$

Notice that the discriminant of the quadratic function inside the brackets must be strictly positive:

$$\Delta = p^2 - 4q > 0,$$

which in turn implies that P_2 must have two distinct real roots. Denoting these two roots by π_{01} and π_{02} , now we can explicitly write out their expressions:

$$\pi_{01}, \pi_{02} = \frac{-p \pm \sqrt{\Delta}}{2}.$$

Furthermore, given Assumption G.3, we have that $P_2(\tilde{m}) > 0$. Since $\widehat{K} < 0$, we can directly conclude that the two roots of P_2 must be located on the opposite sides of $\tilde{m}$. Moreover, since $\pi(t) > \tilde{m}$ must hold, it is obvious that in the case without jumps, the optimal demand for the first stock in the infinite horizon, i.e., π^∞ , is uniquely given by the larger root of P_2 :

$$\pi^\infty = \pi_{01} = \frac{-p + \sqrt{\Delta}}{2}.$$

On the other hand, since $\widehat{K} < 0$ and $P_2(0) = \widehat{K}q > 0$, it is also clear that the smaller root of P_2 , i.e., π_{02} , must be strictly negative. Therefore, we have $\pi_{02} < 0 < \tilde{m} < \pi^\infty$.

In addition, these analytical results also allow us to obtain an explicit representation of the $\mathcal{G}$ function. Specifically, note that we can rewrite the ODE in (G.7) as

$$\frac{d\pi}{dt} = -\frac{\gamma - 1}{\kappa} \widehat{K} \frac{\pi(\pi - \pi^\infty)(\pi - \pi_{02})(\pi - \tilde{m})}{\pi + \frac{\gamma}{\kappa}(\pi - \tilde{m})},$$

then by separation of variables we write

$$\frac{1}{\widehat{K}} \frac{\pi + \frac{\gamma}{\kappa}(\pi - \tilde{m})}{\pi(\pi - \pi^\infty)(\pi - \pi_{02})(\pi - \tilde{m})} d\pi = \overline{K} dt, \quad (\text{G.9})$$

where $\overline{K} := -\frac{\gamma-1}{\kappa} < 0$. Then we define:

$$\mathcal{H}(\pi) := \frac{1}{\widehat{K}} \int \frac{\pi + \frac{\gamma}{\kappa}(\pi - \tilde{m})}{\pi(\pi - \pi^\infty)(\pi - \pi_{02})(\pi - \tilde{m})} d\pi = \overline{K}t + \check{K}, \quad (\text{G.10})$$

where $\mathcal{H}$ denotes the antiderivative of the left-hand side of (G.9), which corresponds to a family of functions that differ by additive constants, whereas $\check{K}$ is a constant to be determined based on the choice of $\mathcal{H}$.

Notice that unlike in Section 2, in the case without jumps it is easy to obtain an explicit representation of the antiderivative $\mathcal{H}$ as follows:²⁷

$$\mathcal{H}(\pi) = \frac{1}{\widehat{K}} [a_1 \ln(\pi - \pi_{02}) + a_2 \ln \pi + a_3 \ln(\pi - \check{m}) + a_4 \ln |\pi - \pi^\infty|], \quad (\text{G.11})$$

for any $\pi \in (\check{m}, \pi^\infty) \cup (\pi^\infty, \infty)$, where

$$\begin{aligned} a_1 &:= \frac{\pi_{02} - \frac{\gamma}{\kappa}(\check{m} - \pi_{02})}{\pi_{02}(\check{m} - \pi_{02})(\pi^\infty - \pi_{02})} > 0, & a_2 &:= \frac{\gamma}{\kappa} \frac{1}{\pi_{02}\pi^\infty} < 0, \\ a_3 &:= -\frac{1}{(\pi^\infty - \check{m})(\check{m} - \pi_{02})} < 0, & a_4 &:= \frac{\pi^\infty + \frac{\gamma}{\kappa}(\pi^\infty - \check{m})}{\pi^\infty(\pi^\infty - \check{m})(\pi^\infty - \pi_{02})} > 0. \end{aligned}$$

It is clear that $\mathcal{H} \in C^1$ on $(\check{m}, \pi^\infty) \cup (\pi^\infty, \infty)$. In addition, note that the $\mathcal{H}$ function in equation (G.11) can also be rewritten as a definite integral plus a constant. Therefore, by analogous arguments for the function $\mathcal{G}$ as in the proof of Proposition 2.14 (c)-(e), it is also straightforward to prove that $\mathcal{H}(\pi) \rightarrow \infty$ for $\pi \nearrow \pi^\infty$ and $\pi \searrow \pi^\infty$ (notice that in the model without jumps, this can be directly seen from equation (G.11), since in either case the term $\frac{a_4}{\widehat{K}} \ln |\pi - \pi^\infty| \rightarrow \infty$ will dominate the value of $\mathcal{H}(\pi)$), $\mathcal{H}'(\pi) > 0$ for all $\pi \in (\check{m}, \pi^\infty)$ and $\mathcal{H}'(\pi) < 0$ for all $\pi \in (\pi^\infty, \infty)$, and $\mathcal{H}$ has two well-defined and continuously differentiable inverses $\mathcal{H}^{-1} \in C^1$ on $\pi \in (\check{m}, \pi^\infty)$ and $\pi \in (\pi^\infty, \infty)$, respectively. Therefore, we can choose the corresponding inverse function according to the value of $\pi(T)$ and obtain the optimal demand for the first stock by $\pi(t) = \mathcal{H}^{-1}(-\overline{K}(T-t) + \mathcal{H}(\pi(T)))$ for $\pi(T) \neq \pi^\infty$ or $\pi(t) = \pi^\infty$ for $\pi(T) = \pi^\infty$, which also satisfies the properties as described in Proposition 2.15.

Finally, verification theorems for the infinite- and finite-horizon cases can also be proven based on the same arguments as we have used in the proofs of Theorems 2.19 and 2.20 under necessary modifications to the definition of the set of admissible strategies (i.e., in Definition 2.17 (e) $\bar{\pi}$ and $\bar{\pi}$ can be arbitrary finite numbers and do not need to be bounded by $\frac{1}{L_1}$ and $\frac{1}{L_2}$, respectively, and the bound M is no longer needed).

H Upper Bound of the Optimal Demand for the Favored Stock

Our baseline model naturally implies a more analytically tractable upper bound on the optimal demand for the favored stock. *Ceteris paribus*, in the absence of jump risk, the agent should be more inclined

²⁷We use the notation $\mathcal{H}$ instead of $\mathcal{G}$ to distinguish the antiderivative we obtain for the case without jumps from the one defined in Section 2 because their definitions are slightly different. Specifically, the function $\mathcal{G}$ is defined by two definite integrals on $[m + \varepsilon_1, \pi^\infty)$ and $(\pi^\infty, \frac{1}{L_1} - \varepsilon_2]$, so the two bounds $m + \varepsilon_1$ and $\frac{1}{L_1} - \varepsilon_2$ contribute to a constant term in the function respectively for each interval. In contrast, the function $\mathcal{H}$ is defined by directly solving an indefinite integral on $(\check{m}, \pi^\infty) \cup (\pi^\infty, \infty)$, where we obtain the antiderivative in an analytical form and omit the constants of integration on each interval without loss of generality (since we already have $\widehat{K}$ on the right-hand side of (G.10)). Technically speaking, if we set $\lambda = \tilde{\lambda} = \hat{\lambda} = 0$ in the definition of $\mathcal{G}$, it will only differ from $\mathcal{H}$ by two additive constants and a slightly smaller domain.

to allocate wealth to risky assets, and particularly to the favored stock, making the no-jump model a natural benchmark for comparison.

The following proposition establishes the corresponding upper bound for our baseline model solution:

Proposition H.1. *(Upper bound of the first stock's optimal demand in the baseline model). For any $\omega = \tilde{\omega} \in (0, \infty)$ fixed, suppose $m < \tilde{m}$, then under Assumptions 2.1, 2.7, 2.11, G.1 and G.3, we have:*

- (a) $\pi^{\infty, J} < \pi^{\infty, NJ}$;
- (b) $\pi^J(T) < \pi^{NJ}(T)$;
- (c) If $\pi^{NJ}(T) \geq \pi^{\infty, J}$ and $\pi^J(T) \leq \pi^{\infty, NJ}$, then $\pi^J(t) < \pi^{NJ}(t)$ for all $t \in [0, T]$;
- (d) If $\pi^{NJ}(T) < \pi^{\infty, J}$ and the following condition holds:

$$\sup_{\pi \in (\pi^{NJ}(T), \pi^{\infty, J})} \chi(\pi) \leq \frac{\hat{\chi}(\pi^{NJ}(T))}{\pi^{\infty, J} P_2(\pi^{NJ}(T))} \quad \text{or} \quad \inf_{\pi \in (\pi^{NJ}(T), \pi^{\infty, J})} \chi(\pi) \geq \frac{\hat{\chi}(\pi^{\infty, J})}{\pi^{NJ}(T) P_2(\pi^{\infty, J})}, \quad (\text{H.1})$$

then $\pi^J(t) < \pi^{NJ}(t)$ for all $t \in [0, T]$;

- (e) If $\pi^J(T) > \pi^{\infty, NJ}$ and the following condition holds:

$$\inf_{\pi \in (\pi^{\infty, NJ}, \pi^J(T))} \chi(\pi) \geq \frac{\hat{\chi}(\pi^{\infty, NJ})}{\pi^J(T) P_2(\pi^J(T))},$$

then $\pi^J(t) < \pi^{NJ}(t)$ for all $t \in [0, T]$, where

$$\chi(\pi) := A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi) - (\pi - \tilde{m}) \left[\frac{dA}{d\pi}(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi}(\pi) \right],$$

$\frac{dA}{d\pi}(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} \frac{dB}{d\pi}(\pi)$ is given by equation (A.3), and

$$\begin{aligned} \hat{\chi}(\pi) := & (1 - \rho^2) \gamma \sigma^2 \left[\pi + \frac{\gamma}{\kappa} (\pi - \tilde{m}) \right] \left[\pi - \tilde{m} + \frac{A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi)}{(1 - \rho^2) \gamma \sigma^2} \right] \\ & \left[\frac{D(\pi)}{\gamma - 1} + \frac{B(\pi)^2}{2\gamma \tilde{\sigma}^2} - \tilde{K} \pi \left(A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi) \right) \right]. \end{aligned}$$

The superscript J denotes the baseline model with jumps while the superscript NJ denotes the model with no jumps.

Proposition H.1 states that as long as the solutions of the first stock's optimal demand to the standard problem (i.e., without preferences for stock) with and without jumps are ordered in a “normal” way, i.e., if the agent is willing to invest more in a world without jump risks, then in our model with agent's preference for this stock, the same ordering will still be preserved both for the asymptotic levels and for the terminal levels of its optimal demand. However, the comparison between $\pi^J(t)$ and $\pi^{NJ}(t)$ for any $t \in [0, T)$ is more complicated and depends on the relative curvatures of $\pi(t)$'s trajectories in the

two cases, which can be seen from the additional conditions required in claims (d) and (e) that need to be imposed on the nonlinear functions D , A and B , i.e., the terms that are directly related to the agent's total and marginal utility losses due to her exposures to jump risks.

Remark. Our main focus is on the first stock's optimal demand $\pi(t)$. On the other hand, the optimal consumption-wealth ratio and the second stock's optimal demand are uniquely determined by $\pi(t)$ via the associated first-order conditions, in both cases with and without jumps. A direct comparison of these two additional quantities is technically involved and does not offer more economic insights, and is therefore omitted from the current discussion.

Proof of Proposition H.1. (a) Given that $\pi^{\infty, NJ} > \check{m} > m$, we must either have $\pi^{\infty, NJ} \geq \frac{1}{L_1}$ or have $\pi^{\infty, NJ} \in (m, \frac{1}{L_1})$. If the former is true, then it directly implies that $\pi^{\infty, NJ} > \pi^{\infty, J}$ since $\pi^{\infty, J} < \frac{1}{L_1}$.

Now, suppose the latter is true, then $P(\pi^{\infty, NJ})$ is well-defined. Given the definitions of m and $\check{m}$, we have

$$m = \frac{(\eta - A(m)) - \rho \frac{\sigma}{\tilde{\sigma}} (\tilde{\eta} - B(m))}{(1 - \rho^2) \gamma \sigma^2} = \check{m} - \frac{A(m) - \rho \frac{\sigma}{\tilde{\sigma}} B(m)}{(1 - \rho^2) \gamma \sigma^2}.$$

Therefore, since $m < \check{m}$, we must have $A(m) - \rho \frac{\sigma}{\tilde{\sigma}} B(m) > 0$.

Moreover, given equation (A.3), the function $A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi)$ is monotonically increasing in π , which implies that $A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi) > 0$ for any $\pi \in [m, \frac{1}{L_1})$.

Notice that the function P can be rewritten as

$$P(\pi) = P_2(\pi) - \left[\frac{D(\pi)}{\gamma - 1} + \frac{B(\pi)^2}{2\gamma \tilde{\sigma}^2} - \tilde{K}[A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi)] \pi \right].$$

Recall that $P_2(\pi^{\infty, NJ}) = 0$. Hence, evaluating P at $\pi = \pi^{\infty, NJ}$ gives

$$P(\pi^{\infty, NJ}) = - \left[\frac{D(\pi^{\infty, NJ})}{\gamma - 1} + \frac{B(\pi^{\infty, NJ})^2}{2\gamma \tilde{\sigma}^2} - \tilde{K}[A(\pi^{\infty, NJ}) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi^{\infty, NJ})] \pi^{\infty, NJ} \right] < 0.$$

By Corollary 2.13, this again implies that $\pi^{\infty, NJ} > \pi^{\infty, J}$.

(b) Similarly to (a), given that $\pi^{NJ}(T) > \check{m} > m$, if $\pi^{NJ}(T) \geq \frac{1}{L_1}$, it must follow that $\pi^{NJ}(T) > \pi^J(T)$ since $\pi^J(T) < \frac{1}{L_1}$.

Suppose $\pi^{NJ}(T) \in (m, \frac{1}{L_1})$. Notice that based on the terminal condition (2.20) we have

$$\begin{aligned} \omega^{-\frac{1}{\kappa}} \frac{1 - \alpha}{\alpha} \alpha^{\frac{1}{\kappa}} \pi^J(T)^{-\frac{\kappa \tilde{\alpha}}{\kappa}} &= (1 - \rho^2) \gamma \sigma^2 [\pi^J(T) - \check{m} + \frac{A(\pi^J(T)) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi^J(T))}{(1 - \rho^2) \gamma \sigma^2}] \\ &> (1 - \rho^2) \gamma \sigma^2 [\pi^J(T) - \check{m}], \end{aligned} \tag{H.2}$$

where the inequality holds due to the proof in (a), i.e., $A(\pi) - \rho \frac{\sigma}{\tilde{\sigma}} B(\pi) > 0$ for any $\pi \in [m, \frac{1}{L_1})$.

On the other hand, the terminal condition (G.8) in the case without jumps states that the above inequality will become equality if $\pi^J(T)$ is replaced by $\pi^{NJ}(T)$.

Note that the left-hand side of the inequality (H.2) is strictly decreasing in $\pi^J(T)$, whereas its right-hand side is strictly increasing in $\pi^J(T)$. Therefore, we must have that $\pi^{NJ}(T) > \pi^J(T)$ so that equality can hold for $\pi^{NJ}(T)$.

(c) If $\pi^{NJ}(T) \geq \pi^{\infty,J}$ and $\pi^J(T) \leq \pi^{\infty,NJ}$, then by (a), (b) and Proposition 2.15 (b), (c), (d) as well as their counterparts in the case without jumps, it is clear that in any case $\pi^J(t)$ will fall in a lower interval, whose upper bound must be strictly smaller than the lower bound of the interval where $\pi^{NJ}(t)$ is located, which holds for any $t \in [0, T]$. Therefore, we must have that $\pi^J(t) < \pi^{NJ}(t)$ for any $t \in [0, T]$.

(d) Notice that although we have restricted the time in our model to start from $t = 0$, in fact, the ODEs as presented in Propositions 2.6 and G.2 are only terminal condition problems that impose an upper bound $t = T$ on time. The properties of $\pi(t)$ as proven by Proposition 2.15 also hold for $t \in (-\infty, T)$. Moreover, by Proposition 2.14 (c) it is clear that $\lim_{t \rightarrow -\infty} \pi(t) = \pi^\infty$, which holds for both cases with and without jumps.

Therefore, in order to prove the claim (d), we first extend our time horizon to a sufficiently large interval, i.e., let $\underline{t} \leq 0$ be sufficiently small such that $\pi^J(\underline{t}) < \pi^{NJ}(\underline{t})$ (existence guaranteed by (a) and the fact that $\pi^J(t), \pi^{NJ}(t) \in C^1$ on $[\underline{t}, T]$ as proven by Proposition 2.15). Then, we aim to show that the two continuous curves $\pi^J(t)$ and $\pi^{NJ}(t)$ do not intersect for any $t \in [\underline{t}, T]$, which further implies that they do not intersect on $[0, T] \subseteq [\underline{t}, T]$ and thus preserve the terminal ordering given by (b).

Specifically, we begin by defining a function $\Delta\pi(t) := \pi^J(t) - \pi^{NJ}(t)$ for any $t \in [\underline{t}, T]$. Clearly, $\Delta\pi \in C^1$ and $\Delta\pi(\underline{t}), \Delta\pi(T) < 0$. We need to show that $\Delta\pi(t) < 0$ for all $t \in [\underline{t}, T]$, which can be achieved by the following three steps:

First, we show that if there exists a $t^* \in (\underline{t}, T)$ such that $\Delta\pi(t^*) = 0$, then under condition (H.1) $\frac{d\Delta\pi}{dt}|_{t=t^*}$ must have a determined sign, either always strictly positive given the first inequality or always strictly negative given the second inequality.

Denote the intersected level as $\pi^J(t^*) = \pi^{NJ}(t^*) = \hat{\pi}$. Since in this case $\pi^J(T) < \pi^{NJ}(T) < \pi^{\infty,J} < \pi^{\infty,NJ}$ by (a) and (b), $\pi^J(t) \in (\pi^J(T), \pi^{\infty,J})$ and $\pi^{NJ}(t) \in (\pi^{NJ}(T), \pi^{\infty,NJ})$ for all $t \in (\underline{t}, T)$ by Proposition 2.15, it must hold that $\hat{\pi} \in (\pi^{NJ}(T), \pi^{\infty,J})$, which further implies that $P_2(\hat{\pi}) > 0$.

On the other hand, note that $P(\pi) = P_2(\pi) - \left[\frac{D(\pi)}{\gamma-1} + \frac{B(\pi)^2}{2\gamma\tilde{\sigma}^2} - \tilde{K}\pi (A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi)) \right]$, $\frac{d\Delta\pi}{dt}\big|_{t=t^*}$ can be written as

$$\begin{aligned} & \frac{d\Delta\pi}{dt}\bigg|_{t=t^*} \\ &= -\frac{\gamma-1}{\kappa}\{ \\ & \quad \frac{P_2(\dot{\pi})\dot{\pi}(\dot{\pi}-\check{m}) + \dot{\pi}\{P_2(\dot{\pi})\frac{A(\dot{\pi})-\rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi})}{(1-\rho^2)\gamma\sigma^2} - [\frac{D(\dot{\pi})}{\gamma-1} + \frac{B(\dot{\pi})^2}{2\gamma\tilde{\sigma}^2} - \tilde{K}\dot{\pi}(A(\dot{\pi}) - \rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi}))](\dot{\pi}-\check{m} + \frac{A(\dot{\pi})-\rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi})}{(1-\rho^2)\gamma\sigma^2})\}}{\dot{\pi} + \frac{\gamma}{\kappa}(\dot{\pi}-\check{m}) + \{\frac{\gamma}{\kappa}\frac{A(\dot{\pi})-\rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi})}{(1-\rho^2)\gamma\sigma^2} + \frac{\dot{\pi}}{(1-\rho^2)\gamma\sigma^2}(\frac{dA}{d\pi}(\dot{\pi}) - \rho\frac{\sigma}{\tilde{\sigma}}\frac{dB}{d\pi}(\dot{\pi}))\}} \\ & \quad - \frac{P_2(\dot{\pi})\dot{\pi}(\dot{\pi}-\check{m})}{\dot{\pi} + \frac{\gamma}{\kappa}(\dot{\pi}-\check{m})}\}. \end{aligned} \tag{H.3}$$

Observing equation (H.3), we can see that the two fractions inside the curly brackets share some common terms. In particular, the first (and second) term(s) in the numerator (denominator) of the first fraction is (are) exactly the same as the numerator (denominator) of the second fraction. Therefore, to compare the magnitude of these two fractions, we can directly compare the fraction of the remaining terms in the first fraction with the second fraction. Notice that $P_2(\dot{\pi}) > 0$ and all the other terms in the second fraction are strictly positive as well, and that $A(\dot{\pi}) - \rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi}) > 0$ as proven in (a) and $\frac{dA}{d\pi}(\dot{\pi}) - \rho\frac{\sigma}{\tilde{\sigma}}\frac{dB}{d\pi}(\dot{\pi}) > 0$ by (A.3). Therefore, we obtain that

$$\begin{aligned} & \frac{d\Delta\pi}{dt}\bigg|_{t=t^*} \leq 0 \\ & \Leftrightarrow \frac{\dot{\pi}\{P_2(\dot{\pi})\frac{A(\dot{\pi})-\rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi})}{(1-\rho^2)\gamma\sigma^2} - [\frac{D(\dot{\pi})}{\gamma-1} + \frac{B(\dot{\pi})^2}{2\gamma\tilde{\sigma}^2} - \tilde{K}\dot{\pi}(A(\dot{\pi}) - \rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi}))](\dot{\pi}-\check{m} + \frac{A(\dot{\pi})-\rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi})}{(1-\rho^2)\gamma\sigma^2})\}}{\frac{\gamma}{\kappa}\frac{A(\dot{\pi})-\rho\frac{\sigma}{\tilde{\sigma}}B(\dot{\pi})}{(1-\rho^2)\gamma\sigma^2} + \frac{\dot{\pi}}{(1-\rho^2)\gamma\sigma^2}(\frac{dA}{d\pi}(\dot{\pi}) - \rho\frac{\sigma}{\tilde{\sigma}}\frac{dB}{d\pi}(\dot{\pi}))} \\ & \geq \frac{P_2(\dot{\pi})\dot{\pi}(\dot{\pi}-\check{m})}{\dot{\pi} + \frac{\gamma}{\kappa}(\dot{\pi}-\check{m})} \\ & \Leftrightarrow \chi(\dot{\pi}) \geq \frac{\hat{\chi}(\dot{\pi})}{\dot{\pi}P_2(\dot{\pi})}. \end{aligned}$$

It is easy to prove that $\hat{\chi}(\pi)$ is strictly positive and monotonically increasing in π on $(\pi^{NJ}(T), \pi^{\infty, J})$, because all the bracketed terms are strictly positive, the term $\pi + \frac{\gamma}{\kappa}(\pi - \check{m})$ is linear in π and has a positive slope, the term $\pi - \check{m} + \frac{A(\pi)-\rho\frac{\sigma}{\tilde{\sigma}}B(\pi)}{(1-\rho^2)\gamma\sigma^2}$ is strictly increasing by (A.3), and the term $\frac{D(\pi)}{\gamma-1} + \frac{B(\pi)^2}{2\gamma\tilde{\sigma}^2} - \tilde{K}\pi (A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi))$ is also strictly increasing by the fact that

$$\begin{aligned} & \frac{d}{d\pi} \left[\frac{D(\pi)}{\gamma-1} + \frac{B(\pi)^2}{2\gamma\tilde{\sigma}^2} - \tilde{K}\pi (A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi)) \right] \\ &= (1 - \tilde{K}) \left(A(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}B(\pi) \right) - \tilde{K}\pi \left(\frac{dA}{d\pi}(\pi) - \rho\frac{\sigma}{\tilde{\sigma}}\frac{dB}{d\pi}(\pi) \right) > 0. \end{aligned}$$

In addition, on $(\pi^{NJ}(T), \pi^{\infty, J})$, it is obvious that $P_2(\pi)$ is also strictly positive but strictly decreasing in π . Therefore, we must have that $\inf_{\pi \in (\pi^{NJ}(T), \pi^{\infty, J})} \frac{\hat{\chi}(\pi)}{\pi P_2(\pi)} > \frac{\hat{\chi}(\pi^{NJ}(T))}{\pi^{\infty, J} P_2(\pi^{NJ}(T))}$. Hence, suppose the first inequality as stated in (d) holds, it must hold that $\chi(\hat{\pi}) < \frac{\hat{\chi}(\hat{\pi})}{\hat{\pi} P_2(\hat{\pi})}$ for any $\hat{\pi} \in (\pi^{NJ}(T), \pi^{\infty, J})$, which implies $\frac{d\Delta\pi}{dt}|_{t=t^*} > 0$ always holds.

Similarly, since $\sup_{\pi \in (\pi^{NJ}(T), \pi^{\infty, J})} \frac{\hat{\chi}(\pi)}{\pi P_2(\pi)} < \frac{\hat{\chi}(\pi^{\infty, J})}{\pi^{NJ}(T) P_2(\pi^{\infty, J})}$, the second inequality in (d) suggests that $\chi(\hat{\pi}) > \frac{\hat{\chi}(\hat{\pi})}{\hat{\pi} P_2(\hat{\pi})}$ for any $\hat{\pi} \in (\pi^{NJ}(T), \pi^{\infty, J})$, which implies $\frac{d\Delta\pi}{dt}|_{t=t^*} < 0$ always holds.

Second, we show that no such t^* exists on $(\underline{t}, T)$.

Without loss of generality, suppose $\frac{d\Delta\pi}{dt}|_{t=t^*} < 0$ for any t^* that satisfies $\Delta\pi(t^*) = 0$. The other case where $\frac{d\Delta\pi}{dt}|_{t=t^*} > 0$ can be proven by symmetric arguments. We proceed to prove by contradiction.

Denote the set of $\Delta\pi$'s zeros on $(\underline{t}, T)$ by $\mathcal{T}^*$, i.e., $\mathcal{T}^* := \{t^* \in (\underline{t}, T) | \Delta\pi(t^*) = 0\}$. Suppose $\mathcal{T}^*$ is not empty and we define $\underline{t}^* := \inf \mathcal{T}^*$. Then, following the same arguments as the proof of Proposition 2.12 (b), it is easy to prove that $\underline{t}^* > \underline{t}$ and $\Delta\pi(\underline{t}^*) = 0$. However, since $\frac{d\Delta\pi}{dt}|_{t=\underline{t}^*} < 0$ and $\Delta\pi$ is continuous, there must exist $\delta^* > 0$ such that $\Delta\pi(t) > 0$ for all $t \in (\underline{t}^* - \delta^*, \underline{t}^*)$. By the intermediate value theorem, there must be another zero of $\Delta\pi$ that exists on the interval $(\underline{t}, \underline{t}^*)$ since $\Delta\pi(\underline{t}) < 0$, which contradicts the definition of $\underline{t}^*$. Therefore, $\mathcal{T}^*$ must be an empty set and no such t^* exists.

Third, we show that $\Delta\pi(t) < 0$ for all $t \in [\underline{t}, T]$, and thus for all $t \in [0, T] \subseteq [\underline{t}, T]$.

This is straightforward from the fact that $\Delta\pi$ is continuous on $[\underline{t}, T]$ with no zero on this interval. Suppose $\Delta\pi(\tilde{t}) > 0$ for some $\tilde{t} \in (\underline{t}, T)$, then the intermediate value theorem immediately implies that $\Delta\pi$ must have at least two zeros on $(\underline{t}, \tilde{t})$ and $(\tilde{t}, T)$, respectively, since $\Delta\pi(\underline{t}), \Delta\pi(T) < 0$. This contradicts the fact that $\Delta\pi$ has no zero. Therefore, $\Delta\pi(t) < 0$ must hold for all $t \in [\underline{t}, T]$, and we can finally conclude that $\pi^J(t) < \pi^{NJ}(t)$ for all $t \in [0, T]$.

(e) The claim (e) can be proven by symmetric arguments to those in (d). The only differences are that in this case if $\pi^J(t)$ and $\pi^{NJ}(t)$ intersect at some t^* , it must hold that $\pi^J(t^*) = \pi^{NJ}(t^*) = \hat{\pi} \in (\pi^{\infty, NJ}, \pi^J(T))$, which thus implies $P_2(\hat{\pi}) < 0$, and since $P_2(\pi) \nearrow 0$ as $\pi \searrow \pi^{\infty, NJ}$ by $\pi^{\infty, NJ}$'s definition, the inequality for the supremum of $\chi(\pi)$ cannot hold since it cannot be smaller than or equal to $-\infty$ by the boundedness theorem (i.e., the continuous function $\chi(\pi)$ must be bounded on the closed and bounded interval $[\pi^{\infty, NJ}, \pi^J(T)]$). $\square$