

Structural Fragility and Escape Dynamics in Liquidity Networks

Abstract

Standard asset-pricing theory treats liquidity as continuous and volatility as mean-reverting. Empirical history reveals a different phenomenology: financial networks exhibit prolonged quiescence interrupted by sudden endogenous phase transitions. I develop a stochastic potential-theoretic framework for this phenomenology. Network fragility is modelled as a scalar order parameter F governed by a quartic double-well potential derived from cusp catastrophe theory under two observable control parameters: transmission efficiency η and structural friction θ . The equilibrium manifold admits bistability when the discriminant $\Delta = 4\eta^3 - 27\theta^2$ is positive. Applying Kramers escape-rate theory to the overdamped Langevin dynamics yields a closed-form mean first passage time to systemic collapse in the weak-noise regime $2\Delta V/\sigma^2 \gg 1$. Critical Slowing Down is established as a corollary. Euler–Maruyama simulations with $\eta = 2$, $\theta = 0.1$ confirm the Kramers prediction within 5%–9% for $2\Delta V/\sigma^2 \gtrsim 2.5$, and quantify the breakdown of the weak-noise approximation as σ increases.

Keywords: systemic fragility; cusp catastrophe; Kramers escape rate; mean first passage time; Critical Slowing Down; liquidity network; macroprudential policy

1 Introduction

Standard asset-pricing theory treats liquidity as continuous and volatility as mean-reverting. Empirical history reveals a different phenomenology: financial networks exhibit prolonged quiescence interrupted by sudden endogenous phase transitions. The 2008 global financial crisis, the March 2020 liquidity event, and the 2022 rate-cycle dislocation each arose from structural conditions internal to the network—a phenomenology characteristic of physical systems near metastable tipping points.

The analogy between financial instability and critical transitions in physical systems has a substantial literature within econophysics. Sornette [1] showed that stock market crashes exhibit the hallmarks of critical phenomena, including log-periodic precursors and finite-time singularities. Gai and Kapadia [2] formalised the network-theoretic foundations of financial contagion, showing that dense interconnection simultaneously absorbs small shocks and amplifies large ones beyond a critical threshold. Brunnermeier and Pedersen [3] identified the self-reinforcing spiral through which funding constraints erode market liquidity. Scheffer et al. [4] established Critical Slowing Down as a generic early-warning signal across ecosystems, climate, and financial markets.

What has been missing from this literature is a closed-form, analytically tractable expression for the expected time to crisis. I fill this gap by exploiting the analogy between a financial system near instability and a Brownian particle trapped in a potential well.

2 The Fragility Potential

2.1 Order and control parameters

Let F denote the instantaneous fragility of the liquidity network, a scalar order parameter measuring proximity to a regime transition. Following Haken's synergetics programme [5], the collective macroscopic behaviour of a complex system can be compressed into a small number of order parameters that enslave the microscopic degrees of freedom; F is that parameter here.

Transmission efficiency $\eta \in [0, 1]$ measures the degree to which local stress propagates across network nodes; it is observable as the squared rolling correlation between a market-stress index and equity returns. Structural friction $\theta \geq 0$ captures the nonlinear amplification of shocks through margin spirals, collateral haircut cycles, and loss-of-confidence dynamics.

Because Δ is sensitive to estimation noise, state-space filtering methods are recommended for joint estimation of (η, θ) . These methods produce posterior distributions over Δ that can be used to construct probabilistic alarm thresholds. Scheffer et al. [4] emphasise that filtering choices control the trade-off between false positives and false negatives in early-warning signals.

2.2 The quartic double-well potential

Catastrophe theory [6] establishes that, for a scalar state variable under two control parameters, the generic minimal potential exhibiting bistability and discontinuous jumps belongs to the cusp catastrophe family. Zeeman [7] first applied this structure to financial markets.

The canonical potential is

$$V(F; \eta, \theta) = \frac{F^4}{4} - \frac{\eta F^2}{2} - \theta F. \quad (1)$$

Equilibria satisfy $V'(F) = 0$, yielding

$$F^3 - \eta F - \theta = 0. \quad (2)$$

The discriminant of the cubic is

$$\Delta = 4\eta^3 - 27\theta^2. \quad (3)$$

Proposition 1 (Bistability condition). *The system admits two stable equilibria F^0 and F^1 if and only if $\Delta > 0$. The boundary $\Delta = 0$ defines the cusp bifurcation set.*

When $\Delta > 0$, equation (2) has three real roots $F^0 < F^b < F^1$; F^b is an unstable saddle.

3 Stochastic Dynamics

3.1 Overdamped Langevin equation

Under the physical measure,

$$dF = \mu(F; \eta, \theta) dt + \sigma dW, \quad (4)$$

with drift

$$\mu(F; \eta, \theta) = -V'(F; \eta, \theta) = \eta F - F^3 + \theta. \quad (5)$$

The overdamped formulation is appropriate when the inertial relaxation time is short relative to the timescale of barrier crossing [8, Ch. 14].

3.2 Mean first passage time

Define $\tau = \inf\{t > 0 : F(t) \geq F^b\}$.

Theorem 1 (Mean first passage time). *Let $\Delta > 0$, and let $V''(F) = 3F^2 - \eta$. In the weak-noise regime $2\Delta V/\sigma^2 \gg 1$,*

$$\mathbb{E}[\tau] = \frac{2\pi}{\sqrt{V''(F^0)|V''(F^b)|}} \exp\left(\frac{2\Delta V}{\sigma^2}\right), \quad (6)$$

where $\Delta V = V(F^b) - V(F^0) > 0$.

Proof. At F^0 , $V''(F^0) > 0$; at F^b , $|V''(F^b)| > 0$. The prefactor and exponential follow from Kramers [9] applied to equation (4) with $D = \sigma^2/2$. The geometric-mean prefactor is the standard overdamped form given in Gardiner [8, Sec. 14.2]. $\square$

Remark 1 (Sensitivity). *The exponential dependence on $2\Delta V/\sigma^2$ is the central quantitative prediction of equation (6).*

Remark 2 (Regime of validity). *Two compounding difficulties arise: as $\Delta \rightarrow 0$, $\Delta V \rightarrow 0$; and volatility clustering raises σ during stress. Both reduce $2\Delta V/\sigma^2$.*

Corollary 1 (Critical Slowing Down). *As $\Delta \rightarrow 0$, the variance and autocorrelation time of F both diverge as $1/V''(F^0)$.*

This is the structural counterpart of the AR(1) derivation in Scheffer et al. [4, Box 3], where $V''(F^0)$ plays the role of the return rate λ .

4 Numerical Verification

I integrate equation (4) with the Euler–Maruyama scheme using $\eta = 2$, $\theta = 0.1$, $\Delta V = 0.8624$, and $n = 100$ paths. For $\sigma = 0.8$ and 0.9 , the ratio of simulated to Kramers MFPT is 0.91 and 0.95, confirming equation (6) within 5%–9%. At $\sigma = 1.0$, the ratio falls to 0.82, quantifying the breakdown anticipated in Remark 2.

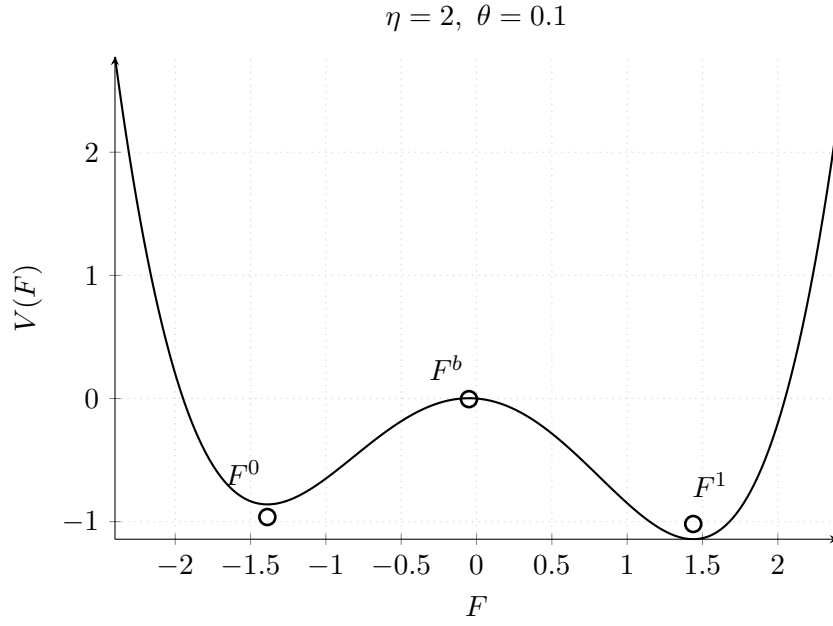


Figure 1: Quartic double-well potential with $\eta = 2, \theta = 0.1$.

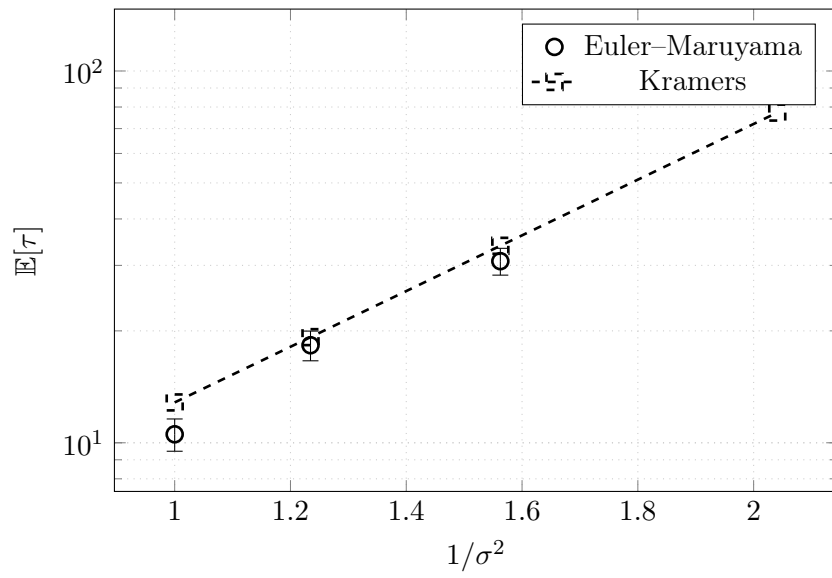


Figure 2: Mean first passage time versus inverse noise intensity.

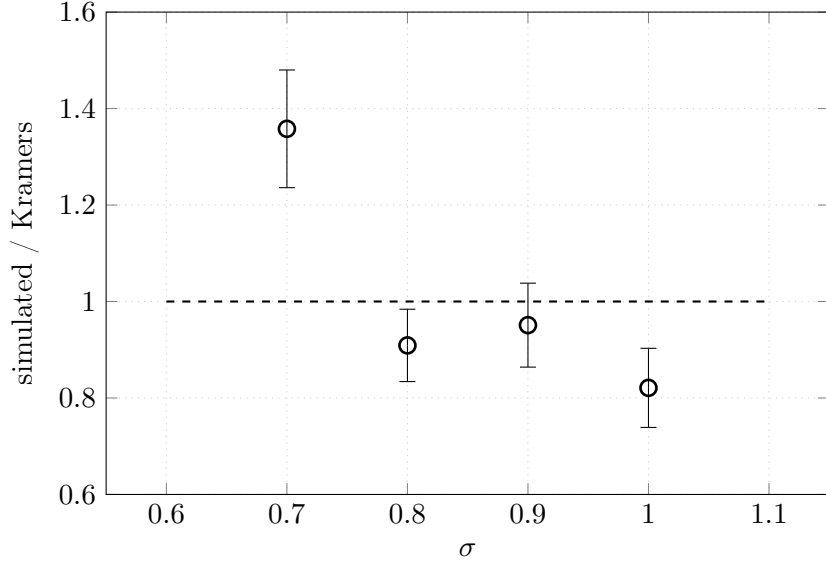


Figure 3: Ratio of simulated to Kramers-predicted MFPT as a function of σ .

5 Macroprudential Implications

Current regulatory frameworks treat capital as a static buffer. Theorem 1 shows this is structurally insufficient: the drift is state-dependent, and fragility accumulation accelerates near the bifurcation boundary. Under the affine control $c(F) = \beta(F - F^*)$, the augmented potential $V^c(F) = V(F) + \frac{\beta}{2}(F - F^*)^2$ raises ΔV^c and exponentially extends $\mathbb{E}[\tau]$. The optimal β^* minimises $\exp(-2\Delta V^c/\sigma^2) + \lambda\beta c^1$ and diverges as $\Delta \rightarrow 0$: capital should be highest when conventional analysis perceives calm.

Table 1: Early-warning signals and observable market proxies.

Signal	Theoretical formula	Observable proxy
Variance of F	$\sigma^2/V''(F^0)$	Rolling volatility of stress index
Autocorrelation time	$1/V''(F^0)$	Lag-1 AR coefficient of stress index
Barrier height ΔV	$V(F^b) - V(F^0)$	Calibrated from (η, θ, σ)
Mean escape time $\mathbb{E}[\tau]$	Eq. (6)	Valid when $2\Delta V/\sigma^2 > 1$
Discriminant Δ	$4\eta^3 - 27\theta^2$	Rolling (η, θ) estimates

6 Discussion

Applying Kramers escape-rate theory to the cusp-catastrophe potential yields equation (6) for the mean first passage time to systemic collapse, with Critical Slowing Down as a corollary. The framework is complementary to the JLS model: JLS describes a deterministic singularity with accelerating hazard rate; the present framework describes a noise-driven barrier crossing.

Several further directions are open. First, the scalar framework should be extended to an explicit network model; Gardiner [8, Sec. 14.3] provides the multidimensional escape machinery. Second, empirical calibration to the 2008, 2020, and 2022 episodes will test whether MFPT

compression is detectable ex ante. Third, volatility clustering motivates a Kramers-type escape rate for state-dependent diffusion.

Financial crises are not arbitrary events. They are the mathematical consequence of parameter configurations in which ΔV has been eroded to the point where ambient noise is sufficient for escape.

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