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SUMMARY: This study introduces a hybrid approach combining Dynamic Bayesian Networks (DBN), Extreme Value Theory (EVT), and copulas to estimate Value at Risk (VaR) for a diversified portfolio of 11 assets. The authors aim to address three major challenges in risk management: volatility spillovers, heavy-tailed innovations, and asymmetric joint dependence. The proposed framework is evaluated on a portfolio of 11 assets and demonstrates interesting empirical properties. At the 90% and 95% confidence levels, all copula-based architectures (Student, Clayton, Survival Gumbel) successfully pass backtesting tests, unlike classical models. At the 99% confidence level, only specific DBN-EVT-Student and DBN-EVT-Clayton copula models maintain valid conditional coverage. These results highlight the need to account for directional volatility spillovers to ensure robust risk forecasts.

METHOD: The study uses a diversified portfolio of 11 assets and evaluates the proposed hybrid approach (DBN-EVT-Copula) against classical models (Variance-Covariance and Historical Simulation). The methodology integrates Dynamic Bayesian Networks, Extreme Value Theory, and copula structures to address the challenges of volatility spillovers, heavy-tailed innovations, and asymmetric joint dependence. The approach is evaluated using backtesting tests for the 90%, 95%, and 99% confidence levels.

KEY FINDINGS:

- All copula-based architectures (Student, Clayton, Survival Gumbel) successfully pass backtesting tests at the 90% and 95% confidence levels.
- Only specific DBN-EVT-Student and DBN-EVT-Clayton copula models maintain valid conditional coverage at the 99% confidence level.
- The proposed hybrid approach demonstrates interesting empirical properties at the 90% and 95% confidence levels.

IMPLICATION FOR TRADING: The findings suggest that incorporating Dynamic Bayesian Networks, Extreme Value Theory, and copulas in VaR estimation can improve the accuracy and robustness of risk forecasts. Practitioners should consider these methods to better account for volatility spillovers, heavy-tailed innovations, and asymmetric joint dependence in their portfolio management strategies. Incorporating these techniques can help in creating more reliable risk models, which is crucial for financial institutions to manage and mitigate risks effectively.

COMBINING DBN-EVT-COPULA FOR PORTFOLIO VaR ESTIMATION

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Abstract

Although the spread of risk across financial markets has been the subject of extensive research, traditional approaches to quantifying VaR often overlook these interactive transmission mechanisms. This study fills this gap by developing the *2-TBN-EVT-Copula* hybrid approach, which integrates dynamic Bayesian networks, extreme value theory (EVT-POT), and copula structures. This framework simultaneously addresses three major challenges in risk management namely volatility spillovers, heavy-tailed innovations, and asymmetric joint dependence. When evaluated on an equally weighted diversified portfolio of 11 assets, the algorithm demonstrates interesting empirical properties. At the 90% and 95% thresholds, all copula-based architectures (Student, Clayton, Survival Gumbel) successfully pass the Kupiec, Independence, and Christoffersen backtesting tests, unlike classical models (Variance-Covariance and Historical Simulation). At the extreme 99% confidence level, only the *DBN-EVT-Student Copula* and *DBN-EVT-Clayton Copula* specifications maintain valid conditional coverage. These results highlight the need to account for directional volatility spillovers to ensure robust risk forecasts.

Keywords : Volatility Spillovers, VaR, Dynamic Bayesian Networks, Extreme Value Theory, Copulas.

Introduction

Over the past few decades, financial markets have been marked by significant volatility, characterized by sharp fluctuations in asset prices, which have at times caused massive losses for financial institutions. Indeed, economic globalization—which followed the explosion of global trade—has created a marked interdependence among markets, whether they be commodity markets, foreign exchange markets, capital markets, or others. Thus, geopolitical tensions such as those observed between Russia and Ukraine, and between the U.S.-Israel bloc and Iran over the past two years, can create price fluctuations in commodity markets and cause uncertainty regarding oil or gold prices, for example.

Furthermore, the growing interconnection of financial markets facilitates the transmission of shocks across different asset classes. Thus, a sudden price change in a given market can trigger speculative behavior affecting prices in another market and thereby influence the volatility of assets in that market. This phenomenon, known as *volatility spillovers* and previously documented by [Beirne et al. \(2013\)](#), highlights the role of financial interdependencies in the global spread of risk.

The increase in financial uncertainty described above has led financial market participants, regulators, and risk managers to continuously develop and improve risk assessment and management methods. One of the methods commonly used in the financial industry is Value at Risk (VaR), popularized by JPMorgan Chase in 1990 as part of the RiskMetrics methodology. VaR can be defined as the maximum loss that an asset or a portfolio of assets may incur over a given time horizon and at a given confidence level. Formally, VaR provides an estimate of the maximum loss under normal market conditions, for a predefined confidence interval. Thus, when a financial institution reports a one-day VaR of \$1,000 at a 95% confidence level, for example, this means that there is a 95% probability that the institution will not lose more than \$1,000 in a single day.

Since its widespread adoption in 1990, VaR has undergone numerous methodological changes aimed at better accounting for the characteristics of financial returns. Indeed, financial time series generally exhibit

stylized facts such as non-normality, volatility clustering, fat tails, leptokurtosis, and asymmetric conditional volatility. These properties challenge the simplifying assumptions of conventional VaR methods. The literature has thus shifted from conventional approaches to estimating VaR—such as nonparametric methods (historical method) and parametric or semiparametric methods such as GARCH models (Angelidis et al., 2004; Restrepo & Isabel, 2012)—to more sophisticated approaches that incorporate the observed properties of financial returns. Among these methods are GARCH-EVT approaches (Bali & Neftci, 2003; McNeil & Frey, 2000; Omari et al., 2017), which account for the fat tails of financial series distributions. Other extensions, such as GARCH-EVT-Copula (Bob, 2013; Patton, 2001; Wang et al., 2010), have been proposed to model the dependence structures among financial assets more realistically. Despite the individual success of these methodologies, we note that none of them takes into account the volatility spillover observed in financial markets. This represents a major gap in the financial risk management literature, particularly in the context of increasingly interconnected financial markets.

The latest advances in probabilistic graph theory offer an alternative for modeling this spread of risk across financial assets. Dynamic Bayesian Networks, in particular, provide a flexible framework for learning the dynamics of conditional dependencies among random variables and for modeling the propagation of risk within a financial system (Ahelegbey, 2016; Neil & Andersen, 2009). In addition to this, there are well-established approaches in risk management. These include Extreme Value Theory, commonly used in the literature to model the distribution tails of random variables, which accounts for the thick distribution tails observed in the returns of financial assets, thereby enabling the modeling of extreme losses (Diebold et al., 2000; A. J. McNeil, 1999). We also have copulas, which take the marginal distributions of a set of random variables as inputs to produce a joint distribution, enabling the modeling of nonlinear dependence structures and tail dependence between assets (Kole et al., 2007).

In light of the findings presented above, this research proposes a novel approach that we call the *DBN-EVT-Copula* approach, in which Dynamic Bayesian Networks, Extreme Value Theory, and Copula models are taken together into a unified framework for estimating daily portfolio VaR. By combining dynamic volatility spillover modeling, accurate tail distribution estimation, and flexible dependence modeling, the proposed approach aims to improve the accuracy and robustness of portfolio VaR forecasts compared to conventional methods.

The main objective of this research is to propose a new approach to estimating VaR by combining Probabilistic Graph Theory, Extreme Value Theory, and Copulas. The innovation of this approach lies in its consideration of volatility spillover to estimate what we call *structural volatility*. That new approach is primarily motivated by the theory of information transmission and financial contagion effects as documented by Beirne et al. (2013) and Cao and Jayasuriya (2012). Indeed, classical financial theory assumes that an asset’s volatility depends only on its own past shocks. However, the literature on the microstructure of global markets highlights that information spreads across assets (Chordia & Subrahmanyam, 2000; King, 1990). A major shock to a sector such as energy, for example, can therefore alter the uncertainty in other sectors, such as transportation or technology.

Although the use of asymmetric GARCH models such as the GJR-GARCH allows to capture the local asymmetric response of an asset’s volatility, it does not incorporate potential shocks to the prices of other assets into the estimation of that asset’s volatility. The use of a Dynamic Bayesian Network therefore serves as a structural correction mechanism in that it allows for the modeling of temporal contagion channels, where the innovation of asset B, for example, at time $t - 1$ becomes a determinant of the volatility of asset A at time t . This approach thus resolves the problems of asymmetric dependence in the distribution tails and cross-sectional transmission of volatility among assets by introducing a three-step semiparametric framework: filtering of autocorrelation using an AR model, a structural re-estimation of conditional volatility via statistical learning using a Dynamic Bayesian Network, and modeling of extreme dependence using Extreme Value Theory and Copulas.

The remainder of the study consists of four sections. The first section is devoted to related work, including the literature on risk measurement using VaR and proposed extensions, as well as a competitor analysis. The next section is devoted to the theoretical framework, in which the mathematics underlying VaR, Dynamic Bayesian Networks, and the DYNOTEARS algorithm, as well as Extreme Value Theory and elements of copula theory, will be presented. The third section focuses on data and the methodology used, and the final section presents the research results and their implications for risk management.

1 Some VaR Models and Contribution of the Study

1.1 Some VaR Modeling Approaches

There are lots of existing literature on VaR estimation, many of which compares traditional parametric or historical methods against approaches designed to capture heavy tails, dependence, or nonlinearity more directly.

With respect to tail modelling, [McNeil and Frey \(2000\)](#) have demonstrated that a two-step GARCH-EVT model produces more accurate one-day VaR forecasts than methods that disregard either the heavy tails of the residual distribution or the time varying nature of volatility. With respect to dependence modelling, GARCH-EVT-Copula frameworks have become increasingly prevalent in recent years. [Zhao et al. \(2023\)](#) combined GARCH, EVT, and copula functions to examine extreme risk spillover from international oil prices into the Chinese stock market, demonstrating that hybrid frameworks of this kind are capable of capturing tail contagion between markets that simpler models fail to detect.

Within the machine learning literature, deep learning models, particularly Long Short-Term Memory (LSTM) networks, have been applied to volatility and VaR forecasting owing to their capacity to learn nonlinear patterns directly from sequential data without requiring a fixed statistical specification. [Nsengiyumva et al. \(2025\)](#) developed a hybrid GARCH-LSTM model for VaR forecasting in the foreign exchange market and found that combining GARCH’s capacity to capture volatility persistence with LSTM’s capacity to capture nonlinear sequential patterns improved VaR accuracy by a meaningful margin relative to either model applied in isolation. These findings support the broader premise underlying the present study, namely that hybrid models combining a classical volatility filtering step with a more flexible second stage tend to outperform models relying on a single technique. The table below presents a concise summary of the trade-offs discussed above across the principal approaches considered in this review.

Table 1: SWOT Analysis

Approach	Strengths	Weaknesses	Opportunities	Threats
Variance-Covariance	Fast, simple, easy to explain	Assumes normal returns, misses fat tails	Good as a quick first-pass check	Understates risk in a crisis
Historical Simulation	No distribution assumption, uses real data	Fully depends on the chosen history window	Works well with long, clean data	Blind to risks not seen before in the sample
Monte Carlo Simulation	Flexible, can use any assumed model	Only as good as the chosen model	Can combine with copulas and EVT	Can look precise while being wrong
EVT (POT)	Built specifically for tail risk	Threshold choice is not automatic	Pairs well with any volatility filter	Needs a long enough sample of extremes
Copula models	Separates marginals from dependence	Choosing the right family takes care	Vine copulas allow mixed dependence types	Wrong family can miss real tail dependence
DBN-EVT-Copula	Explicit, interpretable spillover structure	More complex to build and tune	Can scale to larger richer feature sets	Needs long, clean data OHLC history for every asset

1.2 Contribution of the Research

This study will contribute to the existing literature in three main areas. Firstly, the study will introduce the concept of *structural volatility*, a measure derived from a Dynamic Bayesian Network learned using DYNOTEARS, which captures spillover effects between assets directly and explicitly, rather than assuming a fixed dependence structure or relying solely on an asset’s own historical behaviour. Secondly, the study incorporates three historical volatility estimators, namely Garman-Klass, Rogers-Satchell and Parkinson as candidate feature sets for the network. This helps in enabling a comparison of which volatility measure yields the most informative network structure. Thirdly, the research constructs a complete pipeline extending from structural volatility, through hybrid kernel-EVT tail modelling of standardized

residuals, through copula based joint simulation, to a backtested VaR forecast, evaluated against the two benchmark methods most widely employed within the industry.

Table 2: Competitor Analysis

Study/Method	Spillover Modelled?	Tail Modeled?	Dependence Modelled?	Interpretable?
Variance-Covariance	No	No (normal tails)	Linear Correlation only	Yes
GARCH-EVT	No (own-asset only)	Yes (GPD)	No (single-asset focus)	Partly
GARCH-EVT-Copula	Indirect (via GARCH terms)	Yes (GPD)	Yes (copula)	Can Partly
GARCH-LSTM	No (own-asset only)	Learned implicitly	No	No (black box)
DBN-EVT-Copula	Yes (explicit, learned network)	Yes (hybrid kernel-GPD)	Yes (3 copula families)	Yes (graph is inspectable)

2 Theoretical Framework

2.1 Value at Risk

In this section, we focus on the definition and traditional approaches to calculating VaR. As we noted earlier, VaR is defined as the maximum loss that an asset or portfolio of assets may incur over a given time horizon and for a given confidence level. For this reason, it has become one of the preferred risk measures among financial institutions, as it condenses market risk into a single, readily understandable statistic. The section first presents the mathematical definition and theoretical properties of VaR before reviewing the classical estimation approaches that constitute the benchmark against which the proposed DBN-EVT-Copula framework is evaluated.

2.1.1 Definition and Properties of VaR

Let L denote the random loss of a portfolio over a given investment horizon. The cumulative distribution function of portfolio losses is

$$F_L(l) = P(L \leq l)$$

The VaR for a given confidence level $\alpha \in (0, 1)$ is defined as the lower α quantile of the loss distribution,

$$VaR_\alpha = \inf\{l \in R : F_L(l) \geq \alpha\} \quad (1)$$

Equation (1) implies that

$$P(L > VaR_\alpha) = 1 - \alpha$$

Although VaR is widely used, its mathematical properties should be carefully taken into account. The main properties are monotonicity, positive homogeneity, and translation invariance.

The monotonicity property states that, for two portfolios satisfying $L_1 \leq L_2$, almost surely we have:

$$VaR_\alpha(L_1) \leq VaR_\alpha(L_2) \quad (2)$$

Equation (2) implies that a larger riskier portfolio should necessarily exhibit a larger VaR.

The positive homogeneity property states that for every positive scalar λ ,

$$VaR_\alpha(\lambda L) = \lambda VaR_\alpha(L) \quad (3)$$

Equation (3) states that multiplying the size of the portfolio by a positive scalar λ implies multiply the potential loss of the portfolio by the same size λ .

The translation invariance property finally states that for any deterministic amount k , we have

$$VaR_\alpha(L + k) = VaR_\alpha(L) + k. \quad (4)$$

However, VaR faces a major limitation that is its failure to satisfy subadditivity. Specifically, the inequality

$$VaR(L_1 + L_2) \leq VaR(L_1) + VaR(L_2). \quad (5)$$

does not always hold. This means that, in some non-Gaussian settings, diversifying a portfolio may actually increase its VaR.

2.1.2 Classical Approaches for Estimating VaR

Over the past two decades, numerous approaches to estimating VaR have been proposed. Among these approaches are the Historical Simulation approach and the Variance–Covariance method. Because they are simple to develop and implement, these two approaches are the standard methods used in the financial industry and are considered the benchmarks against which new approaches are evaluated (Dalbudak et al., 2017; Thanh, 2017). It is therefore only natural that we focus specifically on presenting these approaches, as they serve as the benchmark for this study, which proposes a new model for estimating VaR.

Historical Simulation Approach

Historical Simulation is a non-parametric approach that estimates future risk directly from historical returns without imposing any assumption on the underlying return distribution. Suppose that we have the historical returns of a portfolio $\{r_1, r_2, \dots, r_n\}$. The empirical distribution function is

$$\hat{F}(r) = \frac{1}{n} \sum_{i=1}^n I(r_i \leq r), \quad (6)$$

where $I(\cdot)$ is the indicator function and r is a given loss threshold that is set to calculate the proportion of historical returns r_i that are less than or equal to that threshold. The historical VaR corresponds to the empirical quantile,

$$VaR_\alpha = Q_{1-\alpha}(\{r_i\}_{i=1}^n) \quad (7)$$

where $Q_{1-\alpha}$ denote the empirical $1 - \alpha$ quantile.

The Historical Simulation naturally captures skewness, kurtosis, nonlinear dependence, and fat tails in the distribution of asset returns. However, it assumes that future market dynamics will resemble past observations, which is not always guaranteed, as there is no way of knowing whether the past behavior of the return distribution will be exactly the same in the future. Moreover, it cannot estimate losses larger than those already observed, which makes it unsuitable for modeling extremely rare events.

Variance Covariance Approach

The Variance–Covariance approach is a parametric approach which assumes that portfolio returns follow a multivariate normal distribution. Let w the vector of portfolio weights, Σ the variance-covariance matrix of asset returns and μ the vector of expected returns for the portfolio's constituent assets. The portfolio volatility is

$$\sigma_p = \sqrt{w^T \Sigma w}$$

Assuming the normality, the VaR is given by :

$$VaR_\alpha = z_\alpha \sigma_p + \mu_p \quad (8)$$

where μ_p is the expected portfolio return, with

$$\mu_p = w^T \mu$$

The Variance-Covariance method has many advantages: it is computationally efficient, easy to implement, and well-suited for large portfolios. However, its main weakness lies in the underlying Gaussian assumption. In fact, financial returns typically exhibit volatility clustering, skewness, excessive flatness (leptokurtosis), and tail dependence. Consequently, the variance-covariance approach tends to underestimate extreme losses during periods of financial stress.

2.2 Volatility

2.2.1 Concept of Volatility

Volatility is a central concept in finance and risk management. It is defined as the degree of uncertainty associated with asset price movements and is commonly interpreted as the statistical dispersion of returns over a given period. Let P_t be the price of a financial asset at time t . The continuously compounded return is defined as:

$$r_t = \ln \left(\frac{P_t}{P_{t-1}} \right)$$

One of the most commonly used estimators of volatility is historical volatility. It is defined as the standard deviation of the log-return series. Thus, assuming that returns are independently distributed, the unconditional volatility is simply the standard deviation of returns,

$$\sigma = \sqrt{\frac{1}{n-1} \sum_{t=1}^n (r_t - \bar{r})^2}, \quad (9)$$

where

$$\bar{r} = \frac{1}{n} \sum_{t=1}^n r_t.$$

However, empirical evidence shows that the variance of financial returns is not constant. Instead, volatility evolves dynamically over time, exhibiting periods of tranquility alternating with episodes of intense turbulence, a phenomenon commonly referred to as *volatility clustering* (Engle, 1982). Consequently, empirical finance distinguishes several notions of volatility depending on how it is measured or estimated.

2.2.2 Types of Volatility

Several volatility measures have been proposed in the financial literature, each reflecting different sources of market information. The main types are implied volatility, realized volatility and range-based volatility.

Implied volatility is a forward-looking measure extracted from market prices of derivative securities, particularly options. It reflects investors' expectations regarding future market uncertainty and is therefore extensively used in option pricing. Under the Black-Scholes framework, the theoretical option price C satisfies:

$$C = \mathbf{BS}(S, K, T, r, \sigma) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1)$$

with

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where S denotes the underlying asset price, K the strike price, T the time to maturity, r the risk-free rate, σ the volatility parameter and $\Phi(\cdot)$ the CDF of a normal distribution. Since every quantity except σ is directly observable, implied volatility is obtained numerically as:

$$\sigma_{IMP} = \mathbf{BS}^{-1}(C)$$

Realized volatility exploits high-frequency intraday data to estimate the actual variation of asset prices over a trading day. Suppose that we have M intraday observed returns $r_1, r_2, \dots, r_M$. Realized volatility is defined as:

$$RV = \sum_{m=1}^M r_m^2$$

As the sampling frequency increases, the realized volatility converges towards the integrated variance of the price process:

$$RV \rightarrow \int_0^T \sigma_t^2 dt$$

Finally, we have range-based volatility, which is the volatility estimator on which the proposed approach is based. The literature distinguishes four main estimators: the Parkinson estimator, the Garman-Klass estimator, the Rogers-Satchell estimator, and the Yang-Zhang estimator.

2.2.3 Range-Based Volatility

In this part, O_t denote the opening price, H_t the highest observed price of an asset during a trading day, L_t the lowest observed asset price during a trading day, C_t the closing price and T the sample size.

Parkinson Volatility

Parkinson (1980) showed that the daily trading range contains significantly more information about volatility than closing prices alone. The Parkinson variance estimator is given by:

$$\sigma_P^2 = \frac{1}{4T \ln 2} \sum_{t=1}^T \left[\ln \left(\frac{H_t}{L_t} \right) \right]^2.$$

Garman-Klass Volatility

The main limitation of the Parkinson estimator is that it ignores overnight price jumps. To overcome the limitations of Parkinson's estimator, Garman and Klass (1980) incorporated opening and closing prices into the estimation procedure. The Garman-Klass variance estimator is therefore given by:

$$\sigma_{GK}^2 = \frac{1}{T} \sum_{t=1}^T \left[\frac{1}{2} \left(\ln \frac{H_t}{L_t} \right)^2 - (2 \ln 2 - 1) \left(\ln \frac{C_t}{O_t} \right)^2 \right]$$

Rogers-Satchell Volatility

Rogers and Satchell (1991) proposed an estimator specifically designed to remain unbiased in the presence of non-zero drift. Its variance estimator defined as:

$$\sigma_{RS}^2 = \frac{1}{T} \sum_{t=1}^T \left[\ln \left(\frac{H_t}{C_t} \right) \ln \left(\frac{H_t}{O_t} \right) + \ln \left(\frac{L_t}{C_t} \right) \ln \left(\frac{L_t}{O_t} \right) \right].$$

Yang-Zhang Volatility

Yang and Zhang (2000) proposed a range-based volatility estimator by combining three independent sources of variability including overnight volatility, open-to-close volatility and Rogers-Satchell volatility. The estimator is

$$\sigma_{YZ}^2 = \sigma_o^2 + k\sigma_c^2 + (1 - k)\sigma_{RS}^2,$$

where

$$\sigma_o^2 = \frac{1}{T-1} \sum_{t=1}^T \left(\ln \frac{O_t}{C_{t-1}} - \bar{o} \right)^2, \quad \sigma_c^2 = \frac{1}{T-1} \sum_{t=1}^T \left(\ln \frac{C_t}{O_t} - \bar{c} \right)^2,$$

represents overnight volatility and the open-to-close volatility respectively, with,

$$\bar{o} = \frac{1}{T} \sum_{t=1}^T \ln \left(\frac{O_t}{C_{t-1}} \right), \quad \bar{c} = \frac{1}{T} \sum_{t=1}^T \ln \left(\frac{C_t}{O_t} \right), \quad k = \frac{0.34}{1.34 + \frac{T+1}{T-1}}.$$

Methodological Note : The Yang-Zhang estimator was explicitly excluded from this framework. While Yang-Zhang is highly efficient at handling overnight price gaps on equity markets, its mathematical architecture introduces an empirical smoothing effect that replicates the behavior of a stationary GARCH process (Faldziński & Osińska, 2016; Yang & Zhang, 2000). In the context of this study, where the primary objective is to map volatility spillovers via a Dynamic Bayesian Network, such stationary smoothing

would artificially dampen the directional transmission of shocks across time tranches. So by restricting the inputs to Garman-Klass, Rogers-Satchell, and Parkinson volatility estimators, we preserve the raw, unsmoothed intra-day variance innovations. This ensures that the structural learning algorithm captures directional lead-lag causal dependencies.

2.3 Dynamic Bayesian Network and Structural Volatility Estimation

2.3.1 Dynamic Bayesian Network and Specification of the 2-Time-Slice Bayesian Network

A Bayesian Network represents a joint probability distribution over a set of variables $\mathbf{X} = \{X_1, X_2, \dots, X_d\}$ through a Directed Acyclic Graph $G = (V, E)$ where vertices V denote the variables and directed edges E embody conditional dependencies.

To model time-series dependencies without violating the acyclicity constraint, the BN is extended into a Dynamic Bayesian Network (DBN). A DBN models the probability distribution over an infinite unrolled sequence of time slices. Under the first-order Markov assumption, the joint distribution for a time horizon T is factored as:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1}) \quad (10)$$

To capture the *volatility spillover* and then build an asset *structural volatility*, we specify a 2-TBN framework which sets the maximum lag order to $p = 1$. The use a 2-TBN here is motivated by both theoretical and practical considerations. Specifically, by assuming that volatility follows a first-order Markov process, we imply that the current volatility state is conditionally independent of earlier states given the immediately preceding one. This assumption is consistent with the short-term nature of volatility spillovers documented in financial markets and with the structure of most conditional volatility models such as GARCH (Beirne et al., 2013; Engle, 1982).

Let $\mathbf{X}_t \in R^d$ be the vector of d stationary variables observed at time t . The dynamics of the 2-TBN are defined by a conditional transition dynamic structural model:

$$\mathbf{X}_t = \mathbf{X}_t \mathbf{W} + \mathbf{X}_{t-1} \mathbf{A}_1 + \mathbf{E}_t \quad (11)$$

where $\mathbf{W} \in R^{d \times d}$ is the intra-temporal (contemporaneous) weight matrix, representing instantaneous causal links operating within the exact same date t . The diagonal of $\mathbf{W}$ is strictly constrained to zero ($\mathbf{W}_{ii} = 0$), and $\mathbf{W}$ must form a Directed Acyclic Graph. $\mathbf{A}_1 \in R^{d \times d}$ is the inter-temporal matrix of Lag 1, representing the causal impacts of variables at time $t - 1$ on the system at time t . $\mathbf{E}_t \in R^d$ represents a vector of independent, identically distributed Gaussian noise vectors with $E[E_t] = 0$.

2.3.2 Structural Learning via the DYNOTEARS Framework

To estimate the matrices $\mathbf{W}$ and $\mathbf{A}_1$ from empirical data, we employ the DYNOTEARS (Dynamic NOTEARS) optimization framework formulated by Pamfil et al. (2020).

Given an organized data matrix $\mathbf{X} \in R^{n \times d}$, we define the historical lag matrices $\mathbf{Y}_1 = \mathbf{X}_{t-1}$. The learning task is framed as a continuous constrained optimization problem using a penalized least-squares loss function combined with an algebraic acyclicity characterization:

$$\min_{\mathbf{W}, \mathbf{A}_1} \frac{1}{2n} \|\mathbf{X} - \mathbf{XW} - \mathbf{Y}_1 \mathbf{A}_1\|_F + \lambda_w \|\mathbf{W}\|_1 + \lambda_a \|\mathbf{A}_1\|_1 \quad (12)$$

$$\text{subject to } h(\mathbf{W}) = 0$$

Where $\|\cdot\|_F$ represents the matrix Frobenius norm, $\|\cdot\|_1$ is the L_1 -norm element-wise penalty (Lasso) used to enforce sparsity, suppressing weak spurious correlations and isolating true causal pathways. The hyperparameters λ_w and λ_a control the penalization intensity for intra- and inter-temporal edges, respectively. $h(\mathbf{W}) = 0$ is the smooth, continuous algebraic constraint developed by Zheng et al. (2018), which guarantees that the contemporaneous graph remains strictly acyclic. It is mathematically expressed via the matrix exponential trace function:

$$h(\mathbf{W}) = \text{Tr}[\exp(\mathbf{W} \circ \mathbf{W})] - d$$

where $\circ$ denotes the Hadamard product, and Tr is the matrix trace.

Following the continuous optimization, any edge weight whose absolute value falls below the filtration threshold ω is pruned ie $\mathbf{W}_{ij} = 0$ if $|\mathbf{W}_{ij}| < \omega$. This ensures that only meaningful relationships are retained in the final network structure.

2.3.3 Structural Volatility Estimation

Given that the structure learning algorithm of a 2-TBN utilizes normalized variables to prevent scale differences from biasing arc detection, these equations should be expressed at level scales for the structural volatility estimation purposes. If the normalized coefficients are directly injected onto level data, the structural volatility can mathematically explode or become completely distorted.

So let define $y_{i,t}$ as the target variable at level which is the log-variance $\ln(\sigma_{i,t}^2)$ of asset i at time t , $x_{j,t-1}$ the parent variable at level (lagged by one step), μ_i, σ_i the historical mean and standard deviation of the series y_i and μ_j, σ_j the historical mean and standard deviation of the series x_j . The normalized variables processed by the 2-TBN are written as:

$$\tilde{y}_{i,t} = \frac{y_{i,t} - \mu_i}{\sigma_i} \quad \text{and} \quad \tilde{x}_{j,t-1} = \frac{x_{j,t-1} - \mu_j}{\sigma_j} \quad (13)$$

The structural equation derived from the 2-TBN is:

$$\tilde{y}_{i,t} = \sum_j \gamma_{ji} \tilde{x}_{j,t-1} \quad (14)$$

Let us substitute their definitions back to level scale:

$$\frac{y_{i,t} - \mu_i}{\sigma_i} = \sum_j \gamma_{ji} \left(\frac{x_{j,t-1} - \mu_j}{\sigma_j} \right)$$

By multiplying the entire equation by σ_i and isolating $y_{i,t}$, we obtain:

$$y_{i,t} = \mu_i + \sum_j \gamma_{ji} \frac{\sigma_i}{\sigma_j} (x_{j,t-1} - \mu_j)$$

Expanding the terms, we uncover the final form of the level structural equation:

$$y_{i,t} = \left[\mu_i - \sum_j \left(\gamma_{ji} \frac{\sigma_i}{\sigma_j} \right) \mu_j \right] + \sum_j \left(\gamma_{ji} \frac{\sigma_i}{\sigma_j} \right) x_{j,t-1} \quad (15)$$

To obtain the true level structural equations $y_{i,t} = \beta_{0,i} + \sum \beta_{ji} X_{j,t-1}$, we apply the two filters to the results matrices:

$$\beta_{ji} = \gamma_{ji} \times \frac{\sigma_i}{\sigma_j}$$

$$\beta_{0,i} = \mu_i - \sum_j \beta_{ji} \mu_j$$

2.4 Extreme Value Theory

In this section, we provide a brief overview of the Extreme Value Theory (EVT) and its applications in financial risk management. This theory, which is used in statistics to model the extreme values of random variable distributions, generally relies on two methods: Block Maxima Models and Peaks Over Thresholds models.

Block Maxima Models (BM) are based on the direct modeling of extreme realizations. Thus, starting from a given distribution, its peaks—such as the maximum, the minimum, or certain predefined quantiles—are used to construct an asymptotic distribution, thereby allowing inferences to be made about the behavior of the tails of the initially observed distribution. Peaks Over Thresholds (POT) models, on the other hand, model the *excesses* of the distribution based on a given threshold. However, [McNeil and](#)

Frey (2000) point out that POT makes much more efficient use of extreme data, especially when only a limited number of observations are available, making it more practical for empirical studies. Based on these observations, we favor the POT in this research and therefore focus solely on its presentation. In the following sections, we present the Generalized Pareto Distribution, the underlying distribution used in POT models, and then illustrate the modeling of extreme values using the POT.

2.4.1 Generalized Pareto Distribution

As we said earlier, the Generalized Pareto Distribution (GPD) constitutes the main foundation of the POT approach. The theoretical justification for the GPD stems from the Pickands–Balkema–de Haan theorem, which states that for a broad class of underlying distributions, the conditional distribution of exceedances over a high threshold converges asymptotically to a Generalized Pareto Distribution.

The GPD describes the behavior of observations that are below or above a certain threshold of a random variable. In particular, if we consider a random variable X that follows a Generalized Pareto distribution, then its cumulative distribution function $G_{\gamma,\beta}(x)$ is given by the following equation:

$$G_{\gamma,\beta}(x) = \begin{cases} 1 - \left(1 + \frac{\gamma x}{\beta}\right)^{-1/\gamma}, & \text{if } \gamma \neq 0, \\ 1 - \exp\left(-\frac{x}{\beta}\right), & \text{if } \gamma = 0. \end{cases} \quad (16)$$

where $\beta > 0$ denotes the scale parameter, while γ is the shape parameter controlling the tail behaviour of the distribution. The support of the distribution is given by $x \geq 0$ when $\gamma \geq 0$, whereas $0 \leq x \leq -\beta/\gamma$ when $\gamma < 0$. The corresponding probability density function is:

$$g_{\gamma,\beta}(x) = \frac{1}{\beta} \left(1 + \frac{\gamma x}{\beta}\right)^{-1/\gamma-1}, \quad 1 + \frac{\gamma x}{\beta} > 0. \quad (17)$$

The parameter γ determines the heaviness of the tail and therefore governs the probability of observing extreme losses. In particular when $\gamma > 0$, the distribution exhibits heavy tails characteristic of Pareto-type distributions, implying that extremely large losses remain relatively likely. When $\gamma = 0$, the GPD reduces to the exponential distribution, corresponding to light-tailed phenomena where extreme observations decay exponentially fast. When $\gamma < 0$, the distribution possesses a finite upper endpoint, indicating that observations cannot exceed a certain bound.

Several important statistical properties follow directly from the value of the shape parameter. The mean exists only if $\gamma < 1$ and is given by:

$$E(X) = \frac{\beta}{1-\gamma}, \quad \gamma < 1. \quad (18)$$

Similarly, the variance exists only when $\gamma < 1/2$,

$$V(X) = \frac{\beta^2}{(1-\gamma)^2(1-2\gamma)}, \quad \gamma < \frac{1}{2}. \quad (19)$$

2.4.2 Peaks Over Threshold Approach

Suppose we have a set of asset returns observations given by a random variable X with cumulative distribution function $F(x)$, and u a given high threshold. The POT approach focuses on the exceedances above this threshold denoted by the random variable Y and given by :

$$Y = X - u, \quad X > u$$

The conditional distribution of these exceedances is defined as

$$F_u(y) = P(X - u \leq y \mid X > u), \quad y \geq 0. \quad (20)$$

Applying the definition of conditional probability yields

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}, \quad y \geq 0. \quad (21)$$

Equation (21) characterizes the distribution of excess losses given that the threshold has already been exceeded. Since the parent distribution $F(\cdot)$ is generally unknown, EVT provides an asymptotic approximation through the Pickands–Balkema–de Haan theorem. That theorem states that for a broad class of underlying distributions satisfying mild regularity conditions, the conditional excess distribution converges to a GPD as the threshold becomes sufficiently large,

$$\lim_{u \rightarrow x_F} \sup_{0 \leq y < x_F - u} |F_u(y) - G_{\gamma, \beta}(y)| = 0, \quad (22)$$

where x_F denotes the right endpoint of the parent distribution, while $G_{\gamma, \beta}$ represents the cumulative distribution function of the GPD with shape parameter γ and scale parameter $\beta > 0$. Equation (22) constitutes the theoretical foundation of the POT method, implying that regardless of the unknown parent distribution of financial returns, sufficiently extreme observations can always be approximated by a GPD. Consequently,

$$F_u(y) \approx G_{\gamma, \beta}(y), \quad u \text{ sufficiently large.} \quad (23)$$

For the estimation of the tail distribution which interests us, suppose that among N observations, we have N_u that exceed the selected threshold u . The exceedance probability is estimated by:

$$P(X > u) = \frac{N_u}{N} \quad (24)$$

For any value $x > u$,

$$P(X > x) = P(X > u)P(X > x | X > u). \quad (25)$$

Using the GPD approximation,

$$P(X > x) \approx \frac{N_u}{N} \left(1 + \frac{\gamma(x - u)}{\beta} \right)^{-1/\gamma}. \quad (26)$$

Equation (26) provides a direct estimate of the tail probability beyond the observed sample, allowing reliable extrapolation toward very rare events.

2.5 Copulas Theory

Modeling the joint distribution of financial asset returns is a major challenge in risk management. Traditional multivariate models most often summarize the dependence between assets using Pearson’s correlation coefficient. However, observations of financial market dynamics show that returns exhibit nonlinear dependencies, asymmetric co-movements, and fat tails, especially during market crashes such as the subprime crisis in 2007 or the COVID-19 crisis between 2020 and 2021. Thus, linear correlation alone provides an incomplete description of the dependence structure between financial assets.

The theory of copulas, introduced by Sklar in 1959, offers a flexible framework for modeling the dependencies of random variable distributions, thereby allowing for the asymmetric and nonlinear relationships observed in financial markets. The use of copulas in the approach proposed in this research therefore appears essential for reconstructing the multivariate joint distribution required for portfolio VaR estimation. After presenting the basic definitions and properties of copulas, we then present some examples of copulas, notably the Student, Gumbel and Clayton copulas.

2.5.1 Definitions and Basic Properties

Definition of a Copula

A copula is a d -dimensional joint cumulative distribution function of uniform random variables $(u_1, u_2, \dots, u_d)$ with a range between 0 and 1. A copula function is defined by five main propositions:

- $C(U) = C(u_1, u_2, \dots, u_d)$ where u_i is a uniform marginal distribution;
- $C(u_1, u_2, \dots, u_d) : [0, 1]^d \rightarrow [0, 1]$. This means $C(u_1, u_2, \dots, u_d)$ is defined in a d -dimensional hypercube space between 0 and 1. $C(u_1, u_2, \dots, u_d)$ maps the d -dimensional cubic space to a range between 0 and 1.
- $C(u_1, u_2, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0$ if any component in $C(\cdot)$ is 0;
- $C(1, 1, \dots, u_i, \dots, 1, 1) = u_i$ if all other components except u_i are 1;
- $C(\cdot)$ must increase or stay the same when any of the components increases. $C(\cdot)$ is a non-decreasing function for each component.

From the preceding definitions and properties, a copula can be viewed as a function that combines the marginal distributions of several random variables to construct their joint distribution. This principle naturally leads to two fundamental results in copula theory: the Probability Integral Transform which enables the transformation of arbitrary continuous random variables into uniformly distributed variables on the unit interval and Sklar's theorem that establishes that any multivariate distribution can be uniquely decomposed into its marginal distributions and a copula describing their dependence structure.

Theorem 2.1 (Probability Integral Transform) *Suppose a continuous random variable X with a CDF function F . Then $U = F(X)$ follows a standard uniform distribution,*

$$U \sim U[0, 1]$$

Theorem 2.2 (Sklar Theorem) *Let $G(X_1, X_2, \dots, X_d)$ be the joint cumulative distribution function of continuous random variables $X_1, X_2, \dots, X_d$. Also, $F_1(X_1), F_2(X_2), \dots, F_d(X_d)$ are marginal cumulative distribution functions of $X_1, X_2, \dots, X_n$ respectively. Then, there exists a unique copula function $\mathcal{C}$ such that*

$$G(X_1, X_2, \dots, X_n) = \mathcal{C}(F_1(X_1), F_2(X_2), \dots, F_d(X_d))$$

On the other hand, if we have marginal cumulative distribution functions $F_1(X_1), F_2(X_2), \dots, F_d(X_d)$ for continuous random variables $X_1, X_2, \dots, X_d$ and we also have a copula function $\mathcal{C}$, then $G(X_1, X_2, \dots, X_d)$ is a multivariate joint cumulative function with marginal cumulative distributions $F_1(X_1), F_2(X_2), \dots, F_d(X_d)$.

2.5.2 Copulas Properties

The use of copulas in risk management is widespread in the financial industry due to the many properties they offer. In particular, they allow for the joint modeling of the distribution tails of financial time series, which is essential for representing the extreme losses to which financial institutions may be exposed during times of crisis. Among their key properties, three are of particular interest and are discussed in this section: copula bounds, tail dependence measures, and the invariance property.

Copulas bounds

Hoeffding and Frechet found that a copula always falls between an upper bound and a lower bound. Thus, for any copula $C(u_1, u_2, \dots, u_d) : [0, 1]^d \rightarrow [0, 1]$, we have:

$$W(u_1, \dots, u_d) = \max\{1 - d + \sum_{i=1}^d u_i, 0\} \leq C(u_1, \dots, u_d) \leq \min\{u_1, \dots, u_d\} = M(u_1, \dots, u_d)$$

with W the Hoeffding and Frechet lower bound, and M the Hoeffding and Frechet upper bound.

These bounds play a fundamental role in risk management, as they define the extreme limits of the dependence structure. They thus provide a framework for extreme losses that may affect a portfolio of assets, which is represented by a multivariate distribution $f(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ denote the returns or losses of the various assets comprising the portfolio.

Tail dependence measures

In risk management, the linear correlation coefficient generally fails to quantify the probability of simultaneous extreme events. Copulas, through their asymptotic tail dependence coefficients (λ_L and λ_U), make it possible to measure joint behavior at the extreme limits of the distribution.

To illustrate those coefficients, namely the lower tail dependence coefficient λ_L and the upper tail dependence coefficient λ_U , we take the simply case of two random variables (two assets for example) X_1 and X_2 and a copula C . The lower tail dependence coefficient measures the probability that an asset experiences an extreme loss given that another asset is also suffering a severe contraction:

$$\lambda_L = \lim_{u \rightarrow 0^+} P(X_2 \leq F_2^{-1}(u) \mid X_1 \leq F_1^{-1}(u)) = \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u}$$

If $\lambda_L > 0$, the copula exhibits lower tail dependence, meaning that asset liquidations cluster together during market crashes.

Conversely, the upper tail dependence coefficient quantifies co-movements during extreme market rallies:

$$\lambda_U = \lim_{u \rightarrow 1^-} P(X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)) = \lim_{u \rightarrow 1^-} \frac{1 - 2u + C(u, u)}{1 - u}$$

Invariance Property

The invariance property of copulas states that the underlying copula of a random vector remains completely unchanged when its variables undergo strictly increasing, continuous transformations.

Specifically, suppose we have a random vector $X = (X_1, X_2, \dots, X_d)$ with continuous marginal CDFs $F_1, F_2, \dots, F_d$ and a unique copula C . If we apply any set of strictly monotonically increasing functions $T_1, T_2, \dots, T_d$ to the variables, the new random vector $(T_1(X_1), T_2(X_2), \dots, T_d(X_d))$ will have the exact same copula C .

This property ensures that the copula captures the pure statistical dependence structure of the variables, independent of their scale or measurement units. In risk management for instance, this ensures that transforming raw asset volatilities through non-linear filters (such as EVT margins) leaves the structural copula parameters clean and unwarped.

2.5.3 Used Copulas in the Proposed VaR Model

In this section, we present a few copulas used in the financial industry, focusing specifically on the three main ones used in the new approach proposed here.

Generally speaking, there are four families of copulas: Elliptical copulas, which include the normal copula and Student's t -copula; Archimedean copulas, which include Clayton, Gumbel, and Frank copulas; extreme-value copulas; and Vine copulas. For the purposes of this approach, we have focused on Student, Clayton and Gumbel (Survival Gumbel) copulas. The decision to restrict our modeling to these copula structures stems from a need for geometric and econometric completeness. The Clayton and Survival Gumbel copulas allow us to capture contagion phenomena during market crashes. The Student copula on the other hand offers a robust specification with symmetric heavy tails, which is essential for handling residuals with high kurtosis.

Student Copula

The Student t -copula is constructed from the multivariate Student t -distribution and is capable of capturing symmetric tail dependence. It is parameterized by a correlation matrix Σ and degrees of freedom ν . For the case of two random variables X and Y with a correlation coefficient ρ for example, the Student t -copula function is expressed as:

$$C_{\rho, \nu}^t(u, v) = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left[1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)} \right]^{-\frac{\nu+2}{2}} dx dy$$

where t_ν^{-1} is the inverse cumulative distribution function of a standard univariate Student t -distribution with ν degrees of freedom. The lower and upper tail dependence coefficients are given by:

$$\lambda_L = \lambda_U = 2t_{\nu+1} \left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}} \right) > 0$$

Clayton Copula

The Clayton copula is an asymmetric Archimedean copula. It allows to model strong dependencies in the lower tail while exhibiting zero upper tail dependence. The multivariate Clayton copula is defined as:

$$C_\theta^{\text{Clayton}}(u_1, \dots, u_d) = \left(\sum_{i=1}^d u_i^{-\theta} - d + 1 \right)^{-1/\theta}$$

where $\theta \in (0, \infty)$ governs the degree of dependence. As $\theta \rightarrow 0$, the assets approach complete independence, whereas $\theta \rightarrow \infty$ implies perfect co-movement. The lower and upper tail dependence coefficients are given by:

$$\lambda_L = 2^{-1/\theta} > 0, \quad \lambda_U = 0$$

Survival Gumbel Copula

The standard Gumbel copula exhibits asymptotic dependence in the upper tail ($\lambda_U > 0$) and zero dependence in the lower tail ($\lambda_L = 0$). It is therefore designed to model simultaneous extreme gains (market booms). However, since we are interested in a portfolio from the perspective of risk and extreme losses, the structure of interest to us is a simultaneous crash. To model this, we apply a survival transformation to obtain the Survival Gumbel Copula ($C^{\text{Surv-Gumbel}}$), mathematically defined on the uniform margins u and v by:

$$C^{\text{Surv-Gumbel}}(u, v) = u + v - 1 + C^{\text{Gumbel}}(u', v')$$

with $u' = 1 - u, v' = 1 - v$ and

$$C^{\text{Gumbel}}(u', v') = \exp[-((-\ln u')^\theta + (-\ln v')^\theta)^{\frac{1}{\theta}}]$$

The lower and upper tail dependence coefficients are given by:

$$\lambda_L = 2 - 2^{-1/\theta} > 0, \quad \lambda_U = 0$$

3 Data and Methodology

This section is devoted to presenting the empirical framework and implementation protocol for our hybrid architecture—namely, the 2-TBN-EVT-Copula model—as applied to estimating the daily VaR of a portfolio of financial assets.

3.1 Data

To evaluate the predictive performance of the proposed approach, simulations were conducted on a diversified, equally weighted portfolio consisting of 11 representative assets of the US market, with data coming from Yahoo Finance. This portfolio includes six companies in the technology sector (*Apple, Microsoft, Google, Amazon, Meta, Tesla*), two institutions in the banking sector (*JPMorgan Chase, Bank of America*), a company from the energy sector (*Exxon Mobil*), a stock from the healthcare sector (*Johnson & Johnson*), and an allocation to crypto-assets (*Bitcoin*). The study's time frame is divided into two distinct sub-periods of daily observations: a training sample spanning January 1, 2019, to December 31, 2023, used for parameter estimation, and a test sample spanning January 1, 2024, to March 31, 2026, reserved for regulatory backtesting of VaR. Table 3 below summarizes the descriptive statistics of the log returns for these various assets in the training sample.

A review of table 3 shows that BITCOIN and TESLA are the most volatile assets in the sample, with standard deviations of 3.87% and 3.78% respectively. This high volatility reflects the particularly unstable nature of the cryptocurrency market as well as that of the technology sector, whose valuations have experienced significant fluctuations in recent years, particularly with the rise of artificial intelligence.

Table 3: Descriptive Statistics

Ticker	Count	Mean	Std	Min	25%	50%	75%	Max
AAPL	988.0	0.001226	0.019792	-0.104924	-0.008078	0.001047	0.011826	0.113157
AMZN	988.0	0.000316	0.022518	-0.151398	-0.011132	0.000430	0.011365	0.126949
BAC	988.0	0.000446	0.021517	-0.105779	-0.010780	0.000350	0.010792	0.163786
BTC-USD	988.0	0.001532	0.038739	-0.464730	-0.014478	0.001026	0.018976	0.167104
GOOGL	988.0	0.001023	0.020126	-0.099924	-0.008285	0.000929	0.011081	0.091852
JNJ	988.0	0.000531	0.012415	-0.072003	-0.005452	0.000349	0.006810	0.071773
JPM	988.0	0.000653	0.019169	-0.111232	-0.008398	0.000636	0.009697	0.165620
META	988.0	0.000768	0.028714	-0.306391	-0.011641	0.000488	0.013837	0.209308
MSFT	988.0	0.001357	0.018859	-0.099642	-0.008101	0.001421	0.011631	0.132929
TSLA	988.0	0.000542	0.037779	-0.188450	-0.018147	0.001659	0.019449	0.179327
XOM	988.0	0.000318	0.020760	-0.121423	-0.010708	0.000283	0.011136	0.119442

Source : Authors

Furthermore, BITCOIN and META recorded the largest daily losses, with minimum returns of -46.5% and -30.6% , respectively. These observations highlight the presence of extreme events in financial markets, characterized by fat tails, which the normality assumption for returns struggles to represent adequately.

Finally, all assets exhibit average log returns close to zero, with BITCOIN and MICROSOFT posting the highest averages, at 0.153% and 0.136% , respectively. This result is consistent with the efficient market hypothesis, according to which price movements approximately follow a random walk and expected short-term returns remain low.

3.2 Methodology

3.2.1 Effective 2-TBN-EVT-Copula Algorithm

This section details the practical implementation of the proposed 2-TBN-EVT-Copula hybrid algorithm for estimating conditional VaR. Consider a diversified portfolio consisting of n financial assets, whose vector of historical log returns is denoted by $Y_t = (y_{1,t}, \dots, y_{n,t})^\top$ for $t \in \{1, 2, \dots, T\}$. The modeling and predictive simulation workflow consists of 14 sequential steps:

1. *Historical volatility extraction and return calculation:* For each asset a_i , estimate the time series of historical intraday volatilities using the Garman-Klass (GK), Parkinson (PK), and Rogers-Satchell (RS) estimators, while simultaneously calculating the daily log returns;
2. *Linear filtering of returns:* Estimate an autoregressive AR(1) model on each return series to remove first-order autocorrelation, then extract the linear innovation residuals: $e_{i,t} = y_{i,t} - \hat{\phi}_0 - \hat{\phi}_1 y_{i,t-1}$;
3. *Specification of variance matrices:* Construct three separate databases containing the squares of the standardizable residuals $e_{i,t}^2$ and the log-variances associated with each of the three historical estimators (GK, PK, RS);
4. *Network optimization:* Determine the optimal structure of the 2-TBN model via cross-validation grid search by minimizing the Bayesian Information Criterion (BIC). The historical volatility estimator selected for the next step is the one that minimizes the overall BIC score;
5. *Structural Learning:* Fit the final 2-TBN model to the optimal estimator using the DYNOTEARS algorithm in order to map the matrix of lagged causal relationships (volatility spillovers);
6. *Extraction of structural volatilities:* For each asset a_i , generate the series of conditional structural volatilities $\sigma_{i,t}^{struc}$ derived from the structural equations learned by the network;

7. *Residuals standardization:* Calculate the structural standardized residuals, adjusted for temporal variance dependencies: $z_{i,t} = e_{i,t}/\sigma_{i,t}^{struc}$;
8. *Semi-parametric transformation of margins:* Estimate the cumulative distribution function F_i of each series $z_{i,t}$ using a hybrid approach: a nonparametric gaussian kernel density estimation for the center of the distribution, and a parametric EVT-POT model for the tails. This step generates standardized uniform margins $u_{i,t} \sim \mathcal{U}(0, 1)$ according to the Probability Integral Transform theorem;
9. *Modeling and simulation of joint dependencies:* Estimate the parameters and simulate $S = 10\,000$ replicated uniform observation vectors $u_{i,t}^{sim}$ for each tested copula family (Student, Clayton, Survival Gumbel);
10. *Inversion of simulated margins:* Reconstruct the simulated structural standardized residuals by applying the inverse quantile function: $z_{i,T+1}^{sim,s} = F_i^{-1}(u_{i,T+1}^{sim,s})$ for $s \in \{1, \dots, S\}$;
11. *Projection of variance and innovations:* Estimate the structural volatility $\sigma_{i,T+1}^{struc}$ at horizon $T+1$, then project the corresponding return innovations: $e_{i,T+1}^{sim,s} = z_{i,T+1}^{sim,s} \times \sigma_{i,T+1}^{struc}$;
12. *Generating individual return trajectories:* Reconstruct the future log-return trajectories for each asset at time step $T+1$: $y_{i,T+1}^{sim,s} = \hat{\phi}_0 + \hat{\phi}_1 y_{i,T} + e_{i,T+1}^{sim,s}$;
13. *Portfolio aggregation:* Compute the $S = 10\,000$ simulated returns of the overall portfolio by linear projection onto the allocation weights vector w : $r_{p,T+1}^s = w^\top R_{T+1}^s$, where $R_{T+1}^s = [y_{1,T+1}^{sim,s}, \dots, y_{n,T+1}^{sim,s}]^\top$;
14. *VaR Calculation:* Isolate the empirical quantile of order α from the simulated distribution $\{r_{p,T+1}^s\}_{s=1}^{10\,000}$ to determine the final $VaR_t(\alpha)$ of the portfolio.

In step 3, we chose to model the log-variance rather than the variance directly, in order to ensure the positivity constraint. Indeed, using the logarithm ensures that the estimated variance always remains strictly positive. Furthermore, when optimizing the network in step 4, tabu edges were taken into account. There are two types of these constraints:

- **Contemporary arcs:** A shock $e_{i,t}^2$ or the volatility $\log(\sigma_{i,t}^2)$ of an asset i in the portfolio at time t cannot be used as a predictor of the volatility $\log(\sigma_{j,t}^2)$ of a security j at the same time t ;
- **Impossible child vertices:** Shocks $e_{i,t}^2$ are by nature unpredictable so they cannot be modeled as child vertices in the network.

Finally, in step 8, a hybrid approach combining kernel estimation and EVT-POT is used to estimate the CDF of each of the standardized residuals $z_{i,t}$. Specifically, EVT-POT is used for the tails of the distribution, while Gaussian kernel density estimation is used to model the central part of the $z_{i,t}$ distribution. Kernel density estimation (KDE) is a nonparametric method for estimating the probability density function of a random variable from a finite sample. Given observations $x_1, x_2, \dots, x_n$, KDE is defined as:

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - x_i}{h}\right)$$

where $\hat{f}(x)$ is the estimated density at point x , $K(\cdot)$ is the kernel function, $h > 0$ is the smoothing parameter and n is the number of observations. For the case of a gaussian KDE as used in our approach, the kernel function $K(u)$ is:

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

3.2.2 Backtesting the VaR

To validate a VaR model, we must verify that it neither underestimates nor overestimates empirical risk. This validation relies on a binary sequence called the *violation indicator function* (I_t) for a given day t :

$$I_t = \begin{cases} 1 & \text{if } r_t < -\text{VaR}_t(\alpha) \\ 0 & \text{otherwise} \end{cases} \quad (27)$$

Where r_t represents the portfolio return at day t , and α is the theoretical coverage level. Over an out-of-sample testing window of size T , the total number of observed violations is defined as $N = \sum_{t=1}^T I_t$. To test the validity of the VaR model, two tests are generally used: The Kupiec (1995)'s unconditional coverage test and the Christoffersen (1998)'s independence test.

Kupiec's Test

The Kupiec's test focuses exclusively on whether the empirical violation rate $\hat{p} = N/T$ is statistically equal to the theoretical target rate α . the hypotheses of that test are:

$$\begin{cases} H_0 : p = \alpha \\ H_1 : p \neq \alpha. \end{cases}$$

Under the null hypothesis H_0 that the VaR model is valid, the total number of exceptions N follows a binomial distribution $\mathcal{B}(T, \alpha)$. Kupiec implements a Likelihood Ratio (LR) test based on the binomial odds:

$$LR_{uc} = -2 \ln \left[\frac{L(\alpha)}{L(\hat{p})} \right] = -2 \ln \left[\frac{(1-\alpha)^{T-N} \alpha^N}{(1-\hat{p})^{T-N} \hat{p}^N} \right]$$

Expanding this log-likelihood ratio yields the operational formula:

$$LR_{uc} = 2 \left[N \ln \left(\frac{\hat{p}}{\alpha} \right) + (T - N) \ln \left(\frac{1-\hat{p}}{1-\alpha} \right) \right]$$

According to Wilks' theorem, as the sample size $T \rightarrow \infty$, the LR_{uc} statistic converges asymptotically to a χ^2 distribution with 1 degree of freedom:

$$LR_{uc} \sim \chi^2(1)$$

Christoffersen's test

The Kupiec's test faces a major limitation. That limitation is its inability to detect time-dependent clustering of violations. If all N violations occur consecutively during a market crash, the model is structurally flawed even if the total count matches α . Christoffersen's test evaluates whether a violation occurring today is conditionally dependent on whether a violation occurred yesterday.

The violation sequence I_t is modeled as a first-order Markov chain governed by a transition matrix containing four conditional probabilities:

Table 4: Transition Matrix of violation states

Past State ($t-1$)	Current State (t)	Transition Probability	Observed Count
No Violation (0)	No Violation (0)	$p_{00} = 1 - p_{01}$	N_{00}
No Violation (0)	Violation (1)	p_{01}	N_{01}
Violation (1)	No Violation (0)	$p_{10} = 1 - p_{11}$	N_{10}
Violation (1)	Violation (1)	p_{11}	N_{11}

Source : Authors

The empirical conditional transition probabilities are calculated directly from sample frequencies:

$$p_{01} = \frac{N_{01}}{N_{00} + N_{01}} \quad \text{and} \quad p_{11} = \frac{N_{11}}{N_{10} + N_{11}}$$

The hypotheses of the Christoffersen test are:

$$\begin{cases} H_0 : p_{01} = p_{11} = p \\ H_1 : p_{01} \neq p_{11}. \end{cases}$$

The independence likelihood ratio compares the restricted likelihood under the assumption of independence $L(\hat{p})$ against the unrestricted likelihood of the active Markov chain $L(\hat{\Pi})$:

$$LR_{ind} = -2 \ln \left[\frac{L(\hat{p})}{L(\hat{\Pi})} \right] = 2 \left[\ln L(\hat{\Pi}) - \ln L(\hat{p}) \right]$$

where the maximized log-likelihood of the Markov matrix is:

$$\ln L(\hat{\Pi}) = N_{00} \ln(1 - p_{01}) + N_{01} \ln(p_{01}) + N_{10} \ln(1 - p_{11}) + N_{11} \ln(p_{11})$$

Under the null hypothesis of independent violations, this statistic asymptotically follows a χ^2 distribution with 1 degree of freedom:

$$LR_{ind} \sim \chi^2(1)$$

The Conditional Coverage test (LR_{cc}) provides a simultaneous check for both correct unconditional frequency and the independence of violations over time. The statistical Hypotheses of that combined test are $H_0 : p_{01} = p_{11} = \alpha$.

The joint test statistic can be further decomposed into the sum of the two preceding individual tests:

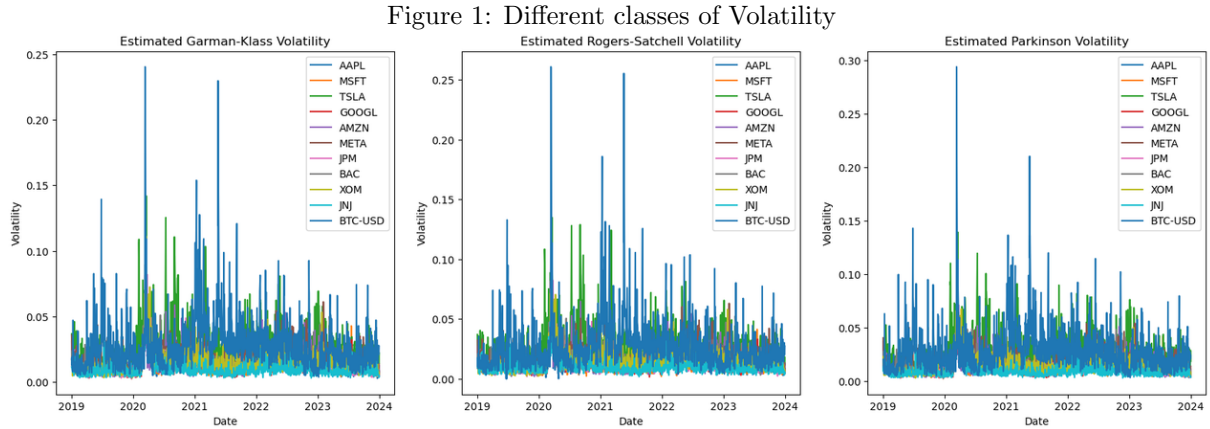
$$LR_{cc} = LR_{uc} + LR_{ind}$$

Because LR_{cc} evaluates two distinct constraints simultaneously (the total rate of exceptions and their temporal distribution), under the null hypothesis of a perfect conditional coverage, the joint statistic converges asymptotically to a χ^2 distribution with 2 degrees of freedom:

$$LR_{cc} \sim \chi^2(2)$$

4 Results

In this section, we present the empirical results of implementing the proposed approach. As we noted earlier, the algorithm was applied to a diversified, equally weighted portfolio of 11 assets from the U.S. capital markets and the cryptoasset market.



The estimates of the three historical volatility classes used—namely, Garman-Klass, Parkinson, and Roger-Satchell—are shown in the figure 1. We can see that all three estimators highlight the same periods of high volatility, particularly during 2020–2021, corresponding to the turbulence associated with the COVID-19 pandemic. The most volatile assets in all three cases are BTC-USD and TSLA, while banking and energy stocks (JPM, BAC, XOM) exhibit more moderate levels of volatility. However, we observe some differences among the various classes. The Parkinson estimator is the most sensitive to intraday fluctuations; the Garman-Klass estimator provides smoother and more efficient estimates under the assumption of no drift; while the Rogers-Satchell estimator offers a robust compromise when assets exhibit price drift. These results show that the choice of estimator influences the estimated level of volatility, without altering the main periods of high or low volatility observed across the various assets.

4.1 Structural Volatility Estimation

Following the first three steps of the algorithm, a network optimization phase was carried out, followed by structural learning. An evaluation of the various estimators reveals that Garman-Klass delivers the best performance, with an overall BIC score of 368.77, thereby outperforming the Rogers-Satchell and Parkinson estimators.

Table 5: Estimated Network parameters

Shock Effects		
Source	Destination	Edge Value
$e_{JPM,t-1}^2$	$\ln(\sigma_{JPM,t}^2)$	0.100
$e_{JPM,t-1}^2$	$\ln(\sigma_{BAC,t}^2)$	0.151
$e_{TSLA,t-1}^2$	$\ln(\sigma_{TSLA,t}^2)$	0.123
$e_{XOM,t-1}^2$	$\ln(\sigma_{XOM,t}^2)$	0.101
Volatility Effects		
Source	Destination	Edge Value
$\ln(\sigma_{AAPL,t-1}^2)$	$\ln(\sigma_{AAPL,t}^2)$	0.258
$\ln(\sigma_{MSFT,t-1}^2)$	$\ln(\sigma_{MSFT,t}^2)$	0.169
$\ln(\sigma_{TSLA,t-1}^2)$	$\ln(\sigma_{TSLA,t}^2)$	0.333
$\ln(\sigma_{GOOGL,t-1}^2)$	$\ln(\sigma_{GOOGL,t}^2)$	0.184
$\ln(\sigma_{AMZN,t-1}^2)$	$\ln(\sigma_{TSLA,t}^2)$	0.115
$\ln(\sigma_{AMZN,t-1}^2)$	$\ln(\sigma_{GOOGL,t}^2)$	0.167
$\ln(\sigma_{AMZN,t-1}^2)$	$\ln(\sigma_{AMZN,t}^2)$	0.362
$\ln(\sigma_{META,t-1}^2)$	$\ln(\sigma_{MSFT,t}^2)$	0.122
$\ln(\sigma_{META,t-1}^2)$	$\ln(\sigma_{AMZN,t}^2)$	0.123
$\ln(\sigma_{META,t-1}^2)$	$\ln(\sigma_{META,t}^2)$	0.271
$\ln(\sigma_{JPM,t-1}^2)$	$\ln(\sigma_{AAPL,t}^2)$	0.119
$\ln(\sigma_{JPM,t-1}^2)$	$\ln(\sigma_{MSFT,t}^2)$	0.128
$\ln(\sigma_{JPM,t-1}^2)$	$\ln(\sigma_{AMZN,t}^2)$	0.109
$\ln(\sigma_{JPM,t-1}^2)$	$\ln(\sigma_{JPM,t}^2)$	0.201
$\ln(\sigma_{JPM,t-1}^2)$	$\ln(\sigma_{BAC,t}^2)$	0.149
$\ln(\sigma_{BAC,t-1}^2)$	$\ln(\sigma_{TSLA,t}^2)$	-0.117
$\ln(\sigma_{BAC,t-1}^2)$	$\ln(\sigma_{BAC,t}^2)$	0.168
$\ln(\sigma_{BAC,t-1}^2)$	$\ln(\sigma_{JNJ,t}^2)$	0.117
$\ln(\sigma_{XOM,t-1}^2)$	$\ln(\sigma_{GOOGL,t}^2)$	0.133
$\ln(\sigma_{XOM,t-1}^2)$	$\ln(\sigma_{JPM,t}^2)$	0.139
$\ln(\sigma_{XOM,t-1}^2)$	$\ln(\sigma_{BAC,t}^2)$	0.111
$\ln(\sigma_{XOM,t-1}^2)$	$\ln(\sigma_{XOM,t}^2)$	0.329
$\ln(\sigma_{JNJ,t-1}^2)$	$\ln(\sigma_{JNJ,t}^2)$	0.222
$\ln(\sigma_{BTC-USD,t-1}^2)$	$\ln(\sigma_{BTC-USD,t}^2)$	0.479

$$\omega^{optimal} = 0.1, \lambda_w^{optimal} = 0.0001, \lambda_a^{optimal} = 0.01$$

The optimal volatility class is the Garman-Klass estimator

Number of Edges: 29

The results of the structural learning, presented in Table 5, clearly highlight risk transmission mechanisms among assets, both through return shocks and volatility effects. First, we observe few shock effects, suggesting that efficiency innovations spread only to a limited extent across assets. In particular, shocks to JPM positively influence the future volatility of JPM (+0.100) as well as that of BAC (+0.151), illustrating risk transmission within the banking sector, as both institutions are highly exposed to common risk factors. Furthermore, we observe that past shocks to TSLA and XOM influence their own volatilities, highlighting the ARCH effects documented by Engle (1982).

Furthermore, the observed volatility effects reveal a much denser network of dependencies. First, we observe strong volatility persistence, illustrated by the presence of autoregressive edges for all assets considered. The estimated coefficients are all positive and range from 0.168 for BAC to 0.479 for BTC-USD. These relationships highlight the phenomenon of *volatility clustering*, which has been extensively documented in the financial literature (Bollerslev, 1986; Cont, 2001; Engle, 1982).

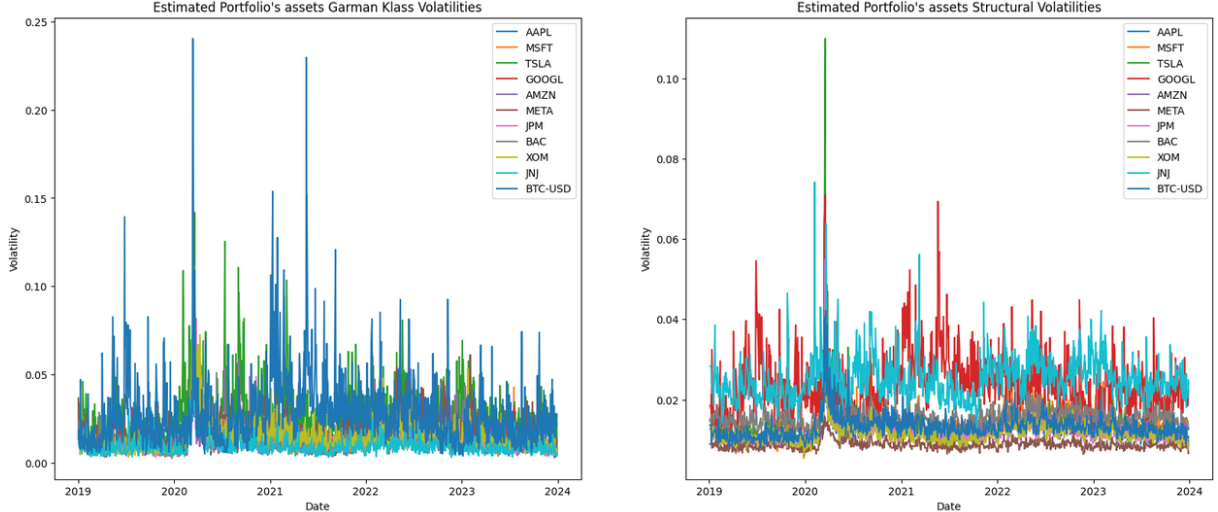
In addition to this individual persistence, the network highlights contagion effects among assets, with companies in the technology sector appearing to be highly interconnected. Thus, AMZN's past volatility positively influences that of TSLA (+0.115) and GOOGL (+0.167), while META's volatility influences MSFT (0.122) and AMZN (0.123). These relationships reflect these companies' shared exposure to the same macroeconomic and sector-specific factors, such as growth expectations, financing conditions, and developments in technology markets.

Furthermore, certain relationships highlight cross-sectoral spillovers. XOM's volatility has a positive effect on that of GOOGL (0.133), JPM (0.139), and BAC (0.111). These results suggest that fluctuations in the energy sector can spread to the technology and financial sectors, likely through macroeconomic conditions, inflation expectations, or changes in commodity prices.

After structural learning, we estimated the structural volatilities $\sigma_{i,t}^{struc}$ for each asset using the coefficients learned by the network. As we noted in the theoretical framework in section 2.3.3, the structure learning algorithm of the 2-TBN uses normalized variables to prevent scale differences from biasing arc detection. Thus, we applied a *denormalization* to the structural equations obtained in the structural learning step to obtain leveled equations. Starting from these new *leveled equations*, the structural volatility $\sigma_{i,t}^{struc}$ of each asset $a_{i,t}$ was estimated recursively.

Figure 2 illustrates the initial Garman-Klass volatilities and the estimated structural volatilities. We can observe that Garman-Klass volatilities exhibit high variability during periods of severe market turbulence, reflecting their sensitivity to instantaneous fluctuations and shocks specific to each asset. In contrast, structural volatilities appear more consistent and less prone to extreme variations, while still capturing the main phases of increased risk. This smoothing of fluctuations results from accounting for structural dependencies among assets during the network's training, which filters out part of the idiosyncratic component of volatility in favor of a measure that is more representative of systemic risk.

Figure 2: Garman-Klass Volatilities vs Estimated Structural Volatilities



Source : Authors

However, despite these differences, both approaches yield a consistent hierarchy of assets in terms of risk, with the most volatile assets in historical estimates also remaining those with the highest levels of structural volatility. Thus, structural volatilities provide a more stable representation of risk dynamics because they incorporate the mechanisms of volatility transmission among the various assets in the portfolio.

4.2 VaR Estimation

Full application of the proposed algorithm to the various copula families yielded the results presented in Table 6. For confidence levels of 90%, 95%, and 99%, the table shows the VaR estimates obtained

from the three families of copulas considered, namely the Student, Survival Gumbel, Clayton, and Vine copulas. Alongside these are also shown the VaR estimates calculated using the variance-covariance method and historical simulations for the aforementioned confidence levels; these two approaches serve as our benchmarks.

Table 6: Estimated VaR (in %) for different methods

Level	Student	Surv Gumbel	Clayton	Var-Covar	Historical
VaR 90%	-1.2198	-1.1114	-1.2468	-1.9240	-1.6028
VaR 95%	-1.6926	-1.4956	-1.8722	-2.4919	-2.4033
VaR 99%	-3.0243	-2.2651	-3.4384	-3.5572	-4.2397

Source : Authors

We observe in Table 6 that for confidence levels of 90% and 95%, the traditional methods yield higher loss estimates than those of the four copulas, with estimated VaRs of -1.924% for the variance-covariance method and -1.603% for historical simulations at 90%, and -2.492% and -2.403% respectively at 95%. The proposed *DBN – EVT – Copula* approach therefore tends to reduce extreme losses compared to traditional approaches, indicating that it provides a more nuanced estimate of risk than traditional methods, which tend to produce more conservative risk measures at intermediate confidence levels.

The previous observations remain virtually the same for a 99% confidence level. However, we observe that the Clayton copula yields a VaR close to that of the variance-covariance method (3.438% versus 3.557%). This result suggests that, in the most extreme loss scenarios, the Clayton copula captures a more pronounced dependence in the lower tail of the return distribution, thereby leading to a more conservative assessment of market risk.

Note that when implementing the algorithm, we used a 95% threshold for each of the asset return series in estimating the CDFs using the semi-parametric Gaussian Kernel-EVT POT approach. Thus, all observations below the 5% quantile and above the 95% quantile are considered extreme realizations and are modeled separately using the EVT-POT, while observations in the interval $[5\% - 95\%]$, considered *central*, were modeled using the Gaussian Kernel Estimation.

4.3 Backtesting the VaR

After estimating the VaR, we proceed to backtesting. To do so, we use standard tests from the financial literature, specifically the Kupiec test and the Christoffersen test. Furthermore, we have opted for static backtesting rather than dynamic backtesting. Indeed, the DBN is specifically designed to model temporal dependencies and continuous regime shifts. Thus, unlike a static model that must be re-estimated over a sliding window to capture new market conditions, the DBN updates its transition probabilities at each time step. Additionally, estimating copula parameters in combination with EVT thresholds and DBN structures requires significant computational power. Forcing dynamic backtesting would introduce instability in parameter estimation and a risk of overfitting, impairing the model’s ability to capture true tail risks.

We start by analyzing the level of violation observed for each model and for different confidence levels over the test period, which, as we noted earlier, covers the period from January 1, 2024, to March 31, 2026, for a total of 587 trading days. We can see from table 7 that at the 90% level, the Student’s copula and the Clayton copula exhibit the best violation rates, with 42 and 41 violations respectively out of 587 daily returns for the portfolio used, representing 9.68% and 9.45% of the observations. The Survival Gumbel copulas show violation rates that are more or less close to the theoretical rate of 10%. In contrast, the historical and variance-covariance methods appear to overestimate the portfolio’s overall risk at the 10% level, with violation rates of 5.06% and 6.91%, respectively.

For the 95% level, however, the model using the Clayton copula performs well, with a violation rate of 5.3%. The Student copula also exhibit violation rates close to the theoretical threshold of 5%. The Survival-Gumbel, historical, and variance-covariance approaches, on the other hand, exhibit violation rates that are far from the theoretical threshold of 5%.

Finally, for 99% VaR, the Student and Clayton copulas, as well as the historical and variance-covariance models, exhibit violation rates very close to the theoretical threshold of 1%. The variance-covariance method (0.92%) also has a violation rate closest to 1% . The Survival copulas, on the other hand, exhibit violation rates that are far from the threshold, indicating an apparent underestimation of risk.

Overall, the results suggest that the *DBN-EVT-Clayton Copula* and *DBN-EVT-Student Copula* methods neither underestimate nor overestimate portfolio risk, regardless of the threshold used. The Survival Gumbel copula, on the other hand, tends to underestimate risk at 95% and 99% confidence levels, while the historical approaches tend to overestimate risk at 90% and 95%. The Kupiec and Christoffersen tests are then used to empirically test the robustness of these results.

Table 7: Violation Rate of the Estimated VaRs on the test data

Method	nviol 90%	vrate 90%	nviol 95%	vrate 95%	nviol 99%	vrate 99%
Student	42	9.68%	26	5.99%	5	1.15%
Survival Gumbel	47	10.83%	33	7.60%	14	3.23%
Clayton	41	9.45%	23	5.30%	4	0.92%
Variance-Covariance	22	5.07%	11	2.53%	4	0.92%
Historical Sim.	30	6.91%	12	2.76%	3	0.69%

Source : Authors

4.3.1 Backtesting the 90% VaR

When evaluating the robustness of the 90% VaR estimates, we observe in table 8 that the proposed DBN-EVT-Copula approach passes all tests regardless of the copula family used, outperforming classical approaches that fail the Kupiec and Christoffersen tests.

The *DBN-EVT-Student Copula* model, with p-values of 82.19%, 98.9%, and 97.5%, respectively, for the Kupiec, independence, and Christoffersen tests, exhibits the highest performance, demonstrating that this model correctly captures the dependency structure and produces reliable risk forecasts at the 90% confidence level. The *DBN-EVT-Survival Gumbel Copula* and *DBN-EVT-Clayton Copula* models also offer interesting statistics regarding risk forecasts. Thus, these results show that the proposed DBN-EVT-Copula approach exhibits a number of violations consistent with expectations (10%) and that these violations are independent over time, indicating that the models are generally well-specified.

In contrast, the historical approaches are generally rejected, as the p-values show that they overestimate risk. Thus, for the 90% VaR, copula-based models—particularly Student or Vine—appear to be the most reliable because they offer the best calibration.

Table 8: Kupiec and Christoffersen tests p values for the 90% VaR

Method	Kupiec's test	Independence test	Christoffersen test
Student	0.8219	0.9898	0.9749
Gumbel	0.5692	0.6087	0.7460
Clayton	0.6986	0.9048	0.9212
Variance-Covariance	0.0002	0.0988	0.0002
Historical Sim.	0.0239	0.1758	0.0312

Source: Authors

4.3.2 Backtesting the 95% VaR

With regard to the 95% VaR, the results in table 9 show that only the *DBN-EVT-Student Copula* and *DBN-EVT-Clayton Copula* models simultaneously satisfy all three validation tests. The *DBN-EVT-Clayton Copula* model offers the best performance, with p-values of 77.7% and 30.0% for the Kupiec and Christoffersen tests, respectively. These results indicate that this model satisfactorily reproduces the frequency of overshoot events as well as their temporal independence, making it the most reliable approach for risk forecasting at the 95% confidence level

Conventional approaches, such as at the 90% level, fail to satisfy the Kupiec and Christoffersen tests. The observed p-values show that the estimated VaR values differ significantly from the theoretical threshold of 5%, clearly illustrating the absence of clustering of extreme losses.

In conclusion, the *DBN-EVT-Clayton Copula* and *DBN-EVT-Student Copula* models outperform classical approaches. The *DBN-EVT-Survival Gumbel Copula* model, however, fails the various tests, indicating that it is not suitable for assessing 95% risk for the data under study.

Table 9: Kupiec and Christoffersen tests p values for the 95% VaR

Method	Kupiec's test	Independence test	Christoffersen test
Student	0.3577	0.2446	0.3329
Gumbel	0.0204	0.3155	0.0410
Clayton	0.7767	0.1272	0.3001
Variance-Covariance	0.0095	0.2715	0.0189
Historical Sim.	0.0200	0.3308	0.0417

Source: Authors

4.3.3 Backtesting the 99% VaR

Finally with regard to the 99% VaR, the results in table 10 show that only the *DBN-EVT-Student Copula* and *DBN-EVT-Clayton Copula* models, alongside the conventional *Variance-Covariance* method, satisfy the global conditional coverage test. The *DBN-EVT-Student Copula* model offers the best and most balanced performance, with p-values of 75.6% and 11.6% for the Kupiec and Christoffersen tests, respectively. These results indicate that this model satisfactorily reproduces the frequency of overshoot events under extreme market stress as well as their temporal independence, making it the most reliable approach for risk forecasting at the 99% confidence level.

Conventional benchmarks, such as the Historical Simulation method, fail to satisfy the independence and Christoffersen tests. The observed p-values (1.1%) highlight a high clustering of extreme losses, showing that non-parametric approaches suffer from critical structural inertia during sequential crisis events.

In conclusion, the *DBN-EVT-Student Copula* and *DBN-EVT-Clayton Copula* models outperform the other structures at this extreme level. The *DBN-EVT-Gumbel Copula* model, however, completely collapses across the various tests, indicating that that model is unsuitable for assessing 99% extreme risk for the data under study.

Table 10: Kupiec and Christoffersen tests p values for the 99% VaR

Method	Kupiec's test	Independence test	Christoffersen test
Student	0.7559	0.0403	0.1163
Gumbel	0.0002	0.4640	0.0008
Clayton	0.8680	0.0228	0.0739
Variance-Covariance	0.8680	0.0228	0.0739
Historical Sim.	0.4936	0.0107	0.0304

Source: Authors

Conclusion

The spread of risk across different types of markets has been the subject of extensive research over the past decade. However, this risk transmission between different asset classes has not always been taken into account in portfolio risk quantification approaches such as VaR. This study therefore fits into this context, namely the introduction of risk diffusion mechanisms across the assets comprising a portfolio into the calculation of VaR. By integrating Dynamic Bayesian Networks with existing approaches—namely EVT-POT and copulas—the new *2-TBN-EVT-Copula* approach addresses three core challenges in modern financial risk management: volatility spillover, heavy-tailed innovations, and asymmetric joint tail

dependence.

Applying this algorithm to an equally weighted portfolio of 11 assets from various economic sectors revealed some interesting empirical properties. First, at the 90% and 95% confidence levels, all copula-based architectures (Student, Survival Gumbel, and Clayton) successfully passed the Kupiec, Independence, and Christoffersen backtesting criteria, whereas the classical benchmarks—namely, Variance-Covariance and Historical Simulation—were rejected due to tail miscalibration. In the case of 99% VaR, the results showed that the *DBN-EVT-Student Copula* and *DBN-EVT-Clayton Copula* passed both classical tests, whereas the classical approaches passed only the Kupiec test. The *DBN-EVT-Survival Gumbel Copula* model proved to be inconsistent, indicating that the Gumbel structure remains unsuitable for mapping extreme market risks, even when its survival transformation is explicitly deployed to capture lower-tail dependence rather than the upper-tail dependence of its standard form.

Ultimately, the study highlights that modeling portfolio tail risk cannot be reduced to a static multivariate distribution problem. By accounting for temporal spillovers through a dynamic Bayesian network prior to modeling extreme tails using EVT and copulas, the proposed model—specifically, the *DBN-EVT-Student Copula*—achieves statistical validity superior to all regulatory horizons. These results therefore highlight the need to account for both directional volatility spillovers and symmetric heavy-tailed joint dependence when developing risk forecasts for interconnected financial markets.

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