

7499384-unwinding-the-crash-generative-diffusion-and-the-geometry-of-order-book-liquidity

SUMMARY: This article surveys recent generative-diffusion models of the limit order book and interprets them through a geometric lens. The order book is seen as a spatially-indexed density of resting volume over price, with depth acting as a diffusion coefficient. The article argues that a liquidity collapse is the approach to a degeneration set, where the induced geometry pinches, and the recovery of liquidity is a reverse-time diffusion. The cost of trading across the collapse is the control energy, which diverges as the degenerate direction is crossed. The article collects empirical evidence from the literature on the 2010 Flash Crash, stylized facts about order-book depth and resilience, and recent diffusion-based order-book simulators. It introduces no new pricing models or trading strategies but offers a vocabulary to connect the order-book literature to the degeneration set.

METHOD: The article uses the geometric lens of degenerate diffusion to interpret empirical data from the order book literature. It reuses notation and results from companion surveys, including the forward SDE, diffusion matrix, induced metric, degeneration set, and generative diffusion representations. The order book is modeled as a spatially-indexed density over price, with depth acting as the diffusion coefficient.

KEY FINDINGS:

- A liquidity collapse is the approach to the degeneration set.
- The recovery of liquidity is a reverse-time diffusion.
- The cost of trading across the collapse is the control energy, which diverges as the degenerate direction is crossed.

IMPLICATION FOR TRADING: The article does not offer new trading strategies or pricing models. Instead, it provides a geometric framework to connect the order book literature to the degeneration set. Practitioners can use this framework to understand the geometric implications of liquidity collapse and recovery, and to analyze the cost of trading across such events. This can inform the design of trading strategies and the interpretation of market data.

Unwinding the Crash: Generative Diffusion and the Geometry of Order-Book Liquidity

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Survey — no new results are claimed; all statements are attributed.

Abstract

We survey recent generative-diffusion models of the limit order book and read them through a single geometric lens: a liquidity collapse is the degeneration of a diffusion. The order book is a density of resting volume over price; its local depth plays the role of a diffusion coefficient, and where depth vanishes the induced geometry pinches in the sense of the companion geometric survey. In that reading a flash crash is the approach to the degeneration set, the reconstruction of liquidity is a reverse-time diffusion or a Schrödinger bridge from a collapsed book to a healthy one, and the cost of trading across the collapse is the control energy that diverges as the degenerate direction is crossed — the same divergence the density-control survey isolates, now read as execution cost. We collect the empirical picture the lens is meant to organize: the forensic literature on the 2010 Flash Crash and liquidity evaporation, the stylized facts of order-book depth and resilience, and the recent body of diffusion-based order-book simulators. The article is expository and, more than its companions, deliberately restrained: the geometry is a lens, not a model. It introduces no pricing model, no trading strategy, and no empirical claim beyond what the cited literature establishes; what it offers is a vocabulary that connects the order-book literature to the pinching picture of the series. It reuses, without re-derivation, the companion surveys *The Geometry of Diffusion*, *From Score to Flow Map*, and *Steering the Fokker–Planck Equation*, and closes the four-survey series at the degeneration set where all of them meet.

1 Introduction

1.1 The crash as a geometric event

A flash crash is a sudden evaporation of liquidity: the resting volume in the order book thins, depth on one side collapses, and price moves violently through a book that can no longer absorb it. The companion surveys of this series have built, in three other settings, a geometric vocabulary for exactly this kind of event — the degeneration of a diffusion, where a direction of the state space loses its noise, the induced metric pinches, and the objects that drive reconstruction become stiff [Ogata 2026a]. This survey asks whether that vocabulary organizes the order-book literature, and argues that it does, at the level of a reading: a liquidity collapse is the approach to the degeneration set, the recovery of liquidity is a diffusion run backward, and the cost of crossing the collapse is the control energy the previous survey showed diverges there.

1.2 Scope, and what this article is not

This is a survey, and of the four it is the one that must be most careful about what it claims. We prove no new theorems and, equally important, we propose no new model: there is no pricing equation, no trading strategy, and no empirical result here beyond what the cited literature already establishes. The geometry of §3 is a lens, not a structural identity — an interpretive frame that relates two existing bodies of work, the order-book literature and the diffusion-geometry of the companion surveys. We state this once here,

develop the empirical picture honestly in §4, and return to the boundary of the claim in a dedicated section (§7), because a reader from quantitative finance is right to be wary of geometric analogy standing in for evidence. Where the reading is suggestive rather than proved — and on the central analogy of §3, it is suggestive — we say so.

1.3 The order book as a field, and the pinching set

The object of §§2–3 is the limit order book read as a spatially-indexed density. Resting volume is a density over price; the rate at which it is eroded and replenished is a local, state-dependent diffusion; and the depth of the book — the volume available near a price — plays the role of the diffusion coefficient whose smallest eigenvalue the geometric survey watches. When depth vanishes on one side, that coefficient degenerates, and the book sits at the pinching set of the series. Sections 4–6 then develop the three readings this makes available: the empirical picture of collapse and of diffusion-based book models (§4), the reconstruction of liquidity as reverse-time diffusion and as a bridge (§5), and the cost of crossing the collapse as a control energy (§6). Section 7 fixes the limits of the lens, and §8 closes the series.

1.4 Notation, and relation to the companion surveys

We keep the notation of the companion surveys and do not re-derive their results. From [Ogata 2026a] we reuse the forward SDE $dX_t = b dt + \sigma dW_t$, the diffusion matrix $a = \sigma \sigma^\top$ with least eigenvalue $\lambda_{\min}(a)$, the induced metric $g = a^{-1}$, and the degeneration set $\Sigma = \{\lambda_{\min}(a) = 0\}$ where the metric pinches and curvature concentrates. From [Ogata 2026b] we reuse the reverse-time and probability-flow representations of generative diffusion and the flow map as the two-time solution operator that book simulators implicitly learn. From [Ogata 2026c] we reuse the Schrödinger bridge and the divergence of control energy near Σ . We write the book as a density and use $a(\cdot)$ for the local diffusion / depth field; a symbol table is in Appendix A, and the order-book-field SDE, with the depth-to-eigenvalue correspondence, is in Appendix B.

2 The order book as a diffusion field

2.1 The limit order book and liquidity

A limit order book records, for an asset, the resting buy and sell orders at each price: the *bid* side below the current price and the *ask* side above it, each a profile of volume across price levels [Gould et al. 2013]. Three features summarize its liquidity. The *spread* is the gap between the best bid and best ask; the *depth* is the volume available at and near the top of the book; and the *resilience* is the speed with which depth is replenished after a trade consumes it [Bouchaud et al. 2018]. Liquidity is high when the spread is tight, the book is deep, and resilience is fast; a liquidity collapse is the simultaneous failure of all three — the spread gaps, depth evaporates, and replenishment stalls. These are the empirical quantities the geometry of §3 will reinterpret.

2.2 A spatially-indexed density

Read the book as a density. At each time, the resting volume is a nonnegative profile over price, which (normalized) is a density p_t on the price axis; the bid and ask sides are its mass below and above the mid-price. Orders arrive, cancel, and execute, eroding the profile where trades occur and rebuilding it where new limit orders rest — a stochastic evolution of the density that the order-book modelling literature has long written as a system of interacting point processes or a stochastic PDE [Cont–Stoikov–Talreja 2010; Bouchaud–Mézard–Potters 2002]. We take only the coarse reading the series needs: the book is a density over price whose evolution is a state-dependent diffusion, with the local intensity of arrival and cancellation setting how fast mass disperses at each price.

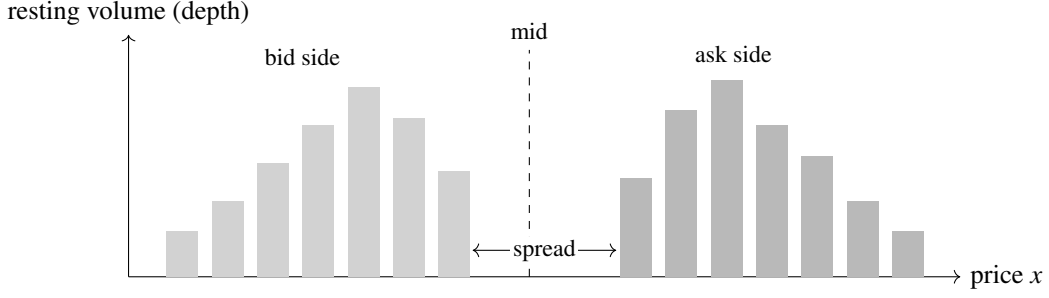


Figure 1: The limit order book as a density over price (§2.2). Resting buy volume (bid side, left) and sell volume (ask side, right) form a profile of depth across price, separated by the spread at the mid-price. Normalized, the profile is a density p_t ; orders arrive, cancel, and execute, evolving it as a state-dependent diffusion whose local rate is set by the depth (§2.3). A liquidity collapse is the evaporation of depth on one side.

2.3 Depth as a diffusion coefficient

The reading that connects the book to the series is this: the local depth sets the local diffusion. Where the book is deep, an incoming order moves the resting profile only slightly — mass disperses slowly, the effective diffusion is small and well-conditioned — and where the book is thin, the same order moves price far, so the effective diffusion in that price direction is large and ill-conditioned, and in the limit of no depth it degenerates. Writing $a(x, t)$ for the local diffusion of the book-density field (Appendix B), the depth on one side controls the eigenvalue of a in that direction, and

$$\text{depth} \rightarrow 0 \iff \lambda_{\min}(a) \rightarrow 0, \quad (2.1)$$

the degeneration the geometric survey studies. The forward direction — erosion of structure into a featureless, noised book — is the noising process of generative diffusion; the reverse direction — reconstruction of a structured, liquid book from a collapsed one — is the reverse-time diffusion of [Ogata 2026b]. This is the bridge to the rest of the survey; (2.1) is the hinge, and it is a reading, made precise as a correspondence in Appendix B, not a derivation of market dynamics from first principles.

3 Liquidity collapse as metric degeneration: the central axis

The axis of this survey is the identification (2.1): a liquidity collapse is the approach to the degeneration set of the companion geometric survey. This section develops that reading and, at its end, flags exactly how far it is meant to carry.

3.1 The pinching set, financially

Recall the degeneration set $\Sigma = \{\lambda_{\min}(a) = 0\}$ where the diffusion loses a direction, the metric $g = a^{-1}$ develops an infinite axis, and the geometry pinches [Ogata 2026a, §7]. Under the depth-to-diffusion reading (2.1), Σ is the set of book states in which depth has vanished on one side: a price direction along which there is no resting volume to absorb flow. On Σ the book cannot diffuse an incoming order — there is nothing for it to move against — and price gaps. The pinching set, financially, is the set of crashed books.

3.2 The crash as approach to Σ

A flash crash is then the dynamic approach to Σ : depth thins on the sell side, the least eigenvalue of the book's diffusion field falls toward zero, the induced metric lengthens without bound in that price direction, and price runs along the lengthening axis — a large move produced by a small flow, because the book has lost the depth that would have absorbed it. The empirical signature of the Flash Crash — a

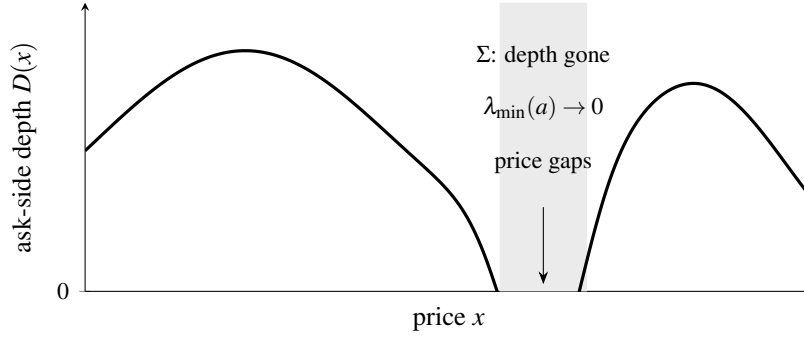


Figure 2: Liquidity collapse as metric degeneration (§3.2). The curve is the ask-side depth $D(x)$ across price; where it falls to zero (shaded), the book has no resting volume to absorb flow. Under the correspondence (2.1), vanishing depth is $\lambda_{\min}(a) \rightarrow 0$: the induced metric $g = a^{-1}$ pinches in that price direction, and price gaps across the hole. This is the order-book face of the degeneration set Σ of the companion geometric survey [Ogata 2026a, §7]; the figure is a schematic of the reading, not data.

violent move on collapsing depth, followed by an equally rapid snap-back once liquidity returned — is, in this reading, a trajectory that approaches Σ and retreats. We develop the empirical side in §4 and the snap-back, as reconstruction, in §5.

3.3 Curvature, read as fragility

The geometric survey showed that near Σ the connection and curvature of g amplify by a^{-1} : the geometry is not merely stretched but increasingly curved as it pinches, and the fields that drive reconstruction become stiff [Ogata 2026a, §§6–7]. Read on the book, this curvature is fragility: near a collapse, small changes in flow produce disproportionately large changes in price, the response of the book to an order is most sensitive exactly where depth is thinnest, and the dynamics are hardest to control. The curvature blow-up of the geometry is the financial intuition that markets are most fragile when liquidity is thinnest, given a precise locus — the neighbourhood of Σ .

3.4 What this reading is, and is not

The identification (2.1) is a lens. It says that two structures — the degeneration of a diffusion geometry and the collapse of order-book liquidity — have the same shape, and that the vocabulary built for one organizes the other. It does *not* say that markets are governed by the specific SDE of the companion survey, that the depth-to-diffusion correspondence is a law rather than a modelling choice, or that any empirical claim about crashes follows from the geometry alone. The correspondence (2.1) is made precise in Appendix B as a statement about a chosen field model of the book, not as a theorem about markets. With that boundary marked, we turn to the empirical literature the lens is meant to organize.

4 The empirical picture

This section collects the empirical and modelling literature that the geometric reading is meant to organize. It is a map, at the level of the companion surveys’ literature sections: we describe what the cited works establish and connect them to the pinching picture, and we do not analyze data or propose a model.

4.1 The Flash Crash and liquidity evaporation

The canonical event is the Flash Crash of 6 May 2010, when U.S. equity-index futures and equities fell and recovered within minutes. The forensic study of Kirilenko, Kyle, Samadi, and Tuzun, using transaction-level audit-trail data on the E-mini S&P 500, established that the crash unfolded as a large

automated sell program met a book whose depth could not absorb it, and that the most active high-frequency intermediaries did not expand their liquidity provision as price fell [Kirilenko et al. 2017]. Easley, López de Prado, and O’Hara documented the run-up to the event as a rise in order-flow toxicity that made liquidity provision unprofitable and drove providers from the book, a “liquidity crash” preceding the price crash [Easley–López de Prado–O’Hara 2011]. The joint regulatory report reached a compatible account of thinning liquidity meeting a large order [CFTC–SEC 2010]. The shared empirical content — depth collapses, the book cannot absorb flow, price gaps, and liquidity returns abruptly — is the trajectory toward and away from Σ of §3.2.

4.2 What the data show about order-book liquidity

Outside the crash, the stylized facts of order-book liquidity are well documented. The limit-order-book survey of Gould et al. collects the empirical regularities of depth, spread, and the shape of the book [Gould et al. 2013]; the microstructure synthesis of Bouchaud, Bonart, Donier, and Gould assembles the statistical laws of trades, quotes, and market impact [Bouchaud et al. 2018]; and the stochastic order-book model of Cont, Stoikov, and Talreja gives a tractable dynamics for depth and its replenishment [Cont–Stoikov–Talreja 2010]. The price impact of a trade — how far a given volume moves price — is, in these literatures, inversely related to depth, the relation Kyle’s model captures through a single liquidity parameter [Kyle 1985; Glosten–Milgrom 1985]. In the reading of §2.3, impact rising as depth falls is exactly the diffusion in a price direction growing as $\lambda_{\min}(a)$ falls: the empirical impact–depth relation is the financial face of the eigenvalue degeneration.

4.3 Generative diffusion models of the order book

A recent and fast-growing literature applies generative models, and specifically diffusion models, to the order book. Early generative book models were autoregressive or adversarial: recurrent-network models of the joint distribution of book transitions [Hultin et al. 2023], conditional GANs that react to market state [Coletta et al. 2021], and GAN models of book volume [Cont et al. 2023]. Diffusion-based simulators followed: DiffVolume generates book-volume snapshots across price levels [Wang–Ventre 2025]; the image-and-inpainting approach of “Painting the Market” generates future book states by treating the book as a structured image [Backhouse et al. 2025]; TRADES builds an order-level diffusion simulator supporting counterfactual analysis [Berti et al. 2025]; and benchmark suites have appeared to compare them [Nagy et al. 2025]. These models learn to noise a book and reconstruct it — the forward and reverse diffusion of [Ogata 2026b] applied to the order-book field — and so they are, in the vocabulary of this series, learning the reconstruction map whose difficulty §5 locates at the collapse.

4.4 What the geometry adds

The empirical literature establishes the facts; the diffusion-model literature builds the generators; and the geometric reading of this series adds an organizing claim about *where* both meet difficulty. The crash literature locates the danger at vanishing depth; the impact literature ties price sensitivity inversely to depth; and the diffusion-book models inherit, by [Ogata 2026b, §7], a reconstruction that is hardest where the book is thinnest — all three are the neighbourhood of Σ . The contribution of the lens is to say that these are one locus, and to import the vocabulary — pinching, curvature, stiff reconstruction, divergent control cost — that the companion surveys developed for it. It is an organizing reading, and it is offered as such.

5 Reconstruction: unwinding the crash

If a crash is the approach to Σ , the recovery of liquidity is the diffusion run backward. This section reads the snap-back of §3.2 through the reconstruction machinery of the companion surveys.

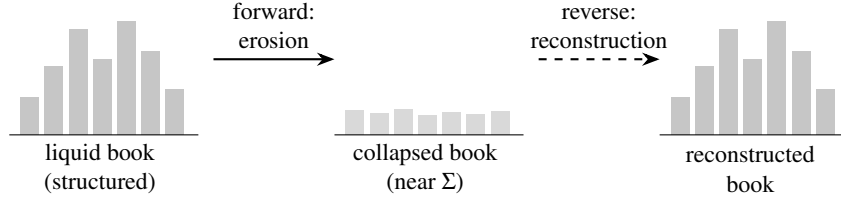


Figure 3: Forward erosion and reverse reconstruction on the book (§5.1). Forward in time, consumption and cancellation flatten a structured, liquid book toward a collapsed one near Σ — the noising process of generative diffusion. Reverse in time, depth re-accumulates and structure returns — the reverse-time diffusion of [Ogata 2026b], which the diffusion-based book simulators of §4.3 implicitly integrate. The Flash Crash snap-back is the reverse step running fast once the trajectory retreats from Σ .

5.1 Reverse-time diffusion on the book

Generation, in the companion survey, reverses a noising process: a structured object is eroded into noise forward in time and reconstructed backward, by integrating a reverse-time SDE or the probability-flow ODE whose only learned ingredient is the score [Ogata 2026b]. Read on the book, the forward erosion is the consumption and cancellation that flattens a structured, liquid book toward a featureless one, and the reverse reconstruction is the return of depth: limit orders re-accumulate, the profile re-forms, and the book regains the structure a crash removed. A diffusion-based book simulator (§4.3) that reconstructs a book from a degraded state is, in this reading, integrating that reverse direction — and the snap-back of the Flash Crash, the abrupt return of liquidity once providers re-entered, is the reconstruction running fast once the trajectory retreats from Σ .

5.2 The bridge reading

The density-control survey gives the two-sided version. Recovering a healthy book from a collapsed one is steering the book-density from a degenerate state back to a liquid target — a Schrödinger bridge between two book states, the full bridge of which the generative reconstruction is the half-bridge [Ogata 2026c, §3]. The optimal restoration drift is the bridge potential’s gradient, and the recovery path is the most likely evolution from collapsed to healthy under the book’s reference dynamics. We draw no quantitative claim from this; it is the bridge vocabulary of the third survey, read on the book, and it places liquidity recovery in the same forward–backward system as generation and control.

5.3 Why reconstruction is hardest at the collapse

Both readings inherit the difficulty the companion surveys locate at Σ . The flow map is stiff at the data boundary, where the score blows up [Ogata 2026b, §7]; the steering rate is lost where the diffusion degenerates [Ogata 2026c, §6]. On the book, this is the statement that reconstruction is hardest exactly at the crash: when depth has fully collapsed, the reverse dynamics that would rebuild the book are most ill-conditioned, recovery is most sensitive to the order in which liquidity returns, and the path back is stiffest. The empirical counterpart is the familiar fragility of a market in the moments after a crash, when the book is thin and a small flow can re-trigger the collapse.

6 The cost of crossing the crash

The control survey added an economic reading of the degeneration set: Σ is not only where reconstruction is hard but where steering is expensive, the control energy diverging as the degenerate direction is crossed [Ogata 2026c, §6.3]. On the book, that control energy has a name — execution cost — and this section makes the connection, at the level of a reading.

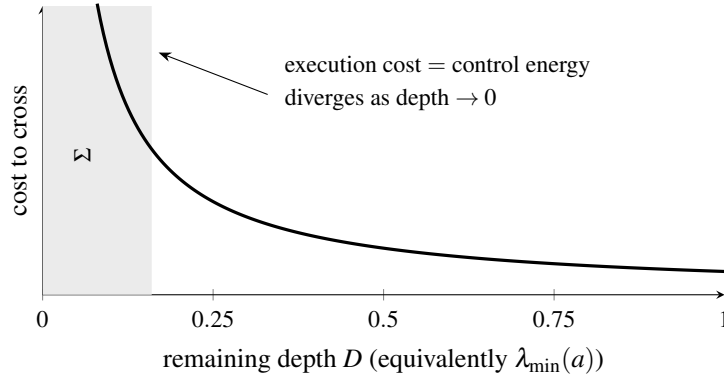


Figure 4: The cost of crossing the crash (§6.2), schematically. The cost of moving price across a region of vanishing depth — the control energy of the density-control survey [Ogata 2026c, §6.2], equivalently the market-impact cost of optimal execution [Almgren–Chriss 2001; Obizhaeva–Wang 2013] — diverges as the remaining depth D , and with it $\lambda_{\min}(a)$, falls to zero on approach to Σ (shaded). The curve is a scaling heuristic illustrating the divergence, not a calibrated impact law; §6.3 marks the boundary of the reading.

6.1 Steering across a degenerate direction

To move price across a region where the book has no depth is to steer the density along a direction in which the diffusion has degenerated, and the density-control survey showed that the energy required to do so diverges as $\lambda_{\min}(a) \rightarrow 0$ [Ogata 2026c, §6.2]. The divergence is the control-theoretic face of the pinching: forcing mass across a direction the noise no longer reaches costs unboundedly much. Read on the book, crossing a collapsed region of price requires absorbing or supplying the depth that is missing, and the cost of doing so grows as the depth that remains shrinks.

6.2 Execution cost as steering cost

This is the order-book quantity the optimal-execution literature studies directly. The cost of executing a large order is set by the market impact it incurs, which rises as the book thins, and the optimal-execution problem — how to schedule a trade to minimize impact cost — is a control problem with exactly the structure of steering a density at minimum energy [Almgren–Chriss 2001; Obizhaeva–Wang 2013]. In the reading of this survey, the execution cost of crossing a thin book is the control energy of §6.1, and its growth as depth vanishes is the divergence of that energy near Σ . The impact literature and the density-control survey describe, in this reading, the same cost: what it takes to move a density through a direction that has lost its depth.

6.3 The boundary of the reading

We flag, as in §3.4, that this is an analogy of structure, not a derivation. The optimal-execution literature has its own models of impact, calibrated to data, and we do not claim the control-energy divergence reproduces any particular impact law or supplies an execution strategy. The claim is only that the two problems share a shape — minimum-energy transport of a density across a degenerate direction — and that the cost diverges where the direction degenerates. The economic reading of Σ is that a crash is expensive to cross in proportion to how completely liquidity has gone, and that is a reading, offered for its organizing value.

7 Position: a lens, not a model

Because this survey carries a geometric analogy into a field with strong empirical standards, it is worth stating plainly what it does and does not claim. The cost of leaving this implicit is that a reader takes the analogy for an assertion about markets, which it is not.

What the survey claims. It claims that the order book admits a reading as a spatially-indexed diffusion field in which local depth plays the role of a diffusion coefficient; that under this reading a liquidity collapse has the same geometric structure as the degeneration of a diffusion — the pinching set of the companion surveys; and that the vocabulary developed there (pinching, curvature as fragility, stiff reconstruction, divergent control cost) organizes three existing bodies of order-book work — the crash literature, the diffusion-book simulators, and the optimal-execution literature — by placing their shared difficulty at one locus. These are claims about *organization* and *analogy*, and the precise version of the central correspondence is the modelling statement of Appendix B.

What the survey does not claim. It does not propose a model of the order book, a law of price impact, or a trading or execution strategy. It makes no empirical claim beyond what the cited literature establishes, runs no data, and calibrates nothing. It does not assert that markets obey the specific SDE of the companion geometric survey, nor that the depth-to-diffusion correspondence is more than a modelling choice that makes the analogy precise. The geometry is a lens for seeing structure shared across literatures; it is not evidence, and it is not a substitute for the models and data of the field it borrows from.

Why the restraint is the point. The series' value, if it has one, is in showing that diffusion-geometric degeneration is a common shape across generative modelling, stochastic control, and market microstructure. That claim is strengthened, not weakened, by refusing to overstate it in the empirical setting where overstatement is easiest to detect. A skeptical reader from quantitative finance should take from this survey a vocabulary, and the burden of any market claim remains with the data and models of the literature it organizes.

8 Outlook: closing the series

The four surveys meet at the degeneration set. The geometric survey built the metric $g = a^{-1}$ and located Σ as the pinching of the diffusion geometry; the generative survey read reversal as the score re-inserted into the drift and met Σ as the stiffness of the flow map at the data boundary; the control survey posed steering as a Schrödinger bridge and met Σ as the loss of the convergence rate and the divergence of control energy; and this survey reads the order book as a diffusion field and meets Σ as the collapse of liquidity, the stiffness of reconstruction, and the cost of crossing a crash.

What unifies the four is what the first survey set out to make available: one diffusion, one metric, and the natural score as the field they share. Generation is the half-bridge, control the full bridge, and liquidity the degenerate direction read economically; transport, reversal, and curvature are three readings of one geometry, and the pinching set is where the difficulty of each concentrates. The series does not claim that this geometry governs any of these fields. It claims that the geometry is a common vocabulary in which their shared structure — and, at Σ , their shared difficulty — becomes legible. That is the whole of the claim, and it is where the series ends.

A Notation

Symbol	Meaning
X_t	price (or log-price) state; reference SDE of [Ogata 2026a]
p_t	book-density: normalized resting volume over price at time t
$b, \sigma, a = \sigma \sigma^\top$	reference drift; dispersion; diffusion / depth field
$\lambda_{\min}(a)$	least eigenvalue of the diffusion field (local depth)
$g = a^{-1}$	induced metric [Ogata 2026a]
$\Sigma = \{\lambda_{\min}(a) = 0\}$	degeneration set; financially, the crashed books
$s_t = \nabla \log p_t$	score of the book-density; reconstruction ingredient
depth	resting volume near the top of book; sets $\lambda_{\min}(a)$ (2.1)
spread	best-ask minus best-bid
resilience	rate of depth replenishment after a trade
impact	price move per unit volume; rises as depth falls

B The order-book field and the depth–eigenvalue correspondence

We make precise the reading (2.1), as a statement about a chosen field model of the book and not as a derivation of market dynamics. Let the book be described by a density $p_t(x)$ of resting volume over price x , evolving by a state-dependent diffusion of the form used in the companion geometric survey,

$$dX_t = b(X_t, t) dt + \sigma(X_t, t) dW_t, \quad a = \sigma \sigma^\top, \quad (\text{B.1})$$

where X_t is the price coordinate and the diffusion field $a(x, t)$ encodes how far a unit of incoming flow displaces the resting profile near price x . The modelling choice that ties this to liquidity is to set the local diffusion inversely to local depth: writing $D(x, t)$ for the depth (resting volume) near price x , take

$$a(x, t) = \kappa D(x, t)^{-1} \Pi(x, t) + a_0, \quad (\text{B.2})$$

with $\kappa > 0$ a scale, Π the projection onto the price direction of thinning depth, and a_0 a small nondegenerate background. The form (B.2) encodes the empirical impact–depth relation of §4.2: where depth is large, a is close to the background a_0 and the book is well-conditioned; where depth falls, the diffusion in the thinning direction grows; and in the limit of vanishing depth,

$$D(x, t) \rightarrow 0 \implies \lambda_{\max}(a) \rightarrow \infty, \quad \lambda_{\min}(g) = \lambda_{\max}(a)^{-1} \rightarrow 0, \quad (\text{B.3})$$

so the metric $g = a^{-1}$ pinches in the price direction of lost depth, and the book state enters the degeneration set Σ of [Ogata 2026a, §7]. This is the correspondence (2.1). Two remarks place it.

1. (A modelling choice.) Equation (B.2) is a choice, not a law: it is the simplest field that reproduces the impact–depth relation and degenerates as depth vanishes. Other functional forms give the same qualitative correspondence; the survey’s reading depends only on the monotone relation $\text{depth} \downarrow \Rightarrow \lambda_{\min}(a) \downarrow$, not on the specific (B.2).
2. (What is borrowed.) Once (B.2) is adopted, everything the companion surveys prove about the neighbourhood of Σ — metric pinching, curvature blow-up, stiff reconstruction, divergent control energy [Ogata 2026a,b,c] — transfers to the book by reading. The transfer is the content of the survey; (B.2) is the bridge that licenses it, and it is a modelling assumption, stated as such.

References

- [1] R. Almgren, N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk* 3(2):5–39, 2001.

- [2] A. Backhouse et al. Painting the market: generative diffusion models for financial limit order book simulation and forecasting. arXiv:2509.05107, 2025.
- [3] A. Berti et al. TRADES: generating realistic market simulations with diffusion models. arXiv preprint, 2025.
- [4] J.-P. Bouchaud, M. Mézard, M. Potters. Statistical properties of stock order books: empirical results and models. *Quantitative Finance* 2(4):251–256, 2002.
- [5] J.-P. Bouchaud, J. Bonart, J. Donier, M. Gould. *Trades, Quotes and Prices: Financial Markets Under the Microscope*. Cambridge University Press, 2018.
- [6] U.S. Commodity Futures Trading Commission and U.S. Securities and Exchange Commission. Findings regarding the market events of May 6, 2010. Joint report, 2010.
- [7] A. Coletta, M. Prata, M. Conti, E. Mercanti, et al. Towards realistic market simulations: a generative adversarial networks approach. *ACM International Conference on AI in Finance (ICAIF)*, 2021.
- [8] R. Cont, S. Stoikov, R. Talreja. A stochastic model for order book dynamics. *Operations Research* 58(3):549–563, 2010.
- [9] R. Cont, M. Cucuringu, J. Kochems, et al. Limit order book simulation with generative adversarial networks. Working paper / SSRN, 2023.
- [10] D. Easley, M. López de Prado, M. O’Hara. The microstructure of the “Flash Crash”: flow toxicity, liquidity crashes, and the probability of informed trading. *Journal of Portfolio Management* 37(2):118–128, 2011.
- [11] L. R. Glosten, P. R. Milgrom. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics* 14(1):71–100, 1985.
- [12] M. D. Gould, M. A. Porter, S. Williams, M. McDonald, D. J. Fenn, S. D. Howison. Limit order books. *Quantitative Finance* 13(11):1709–1742, 2013.
- [13] H. Hultin, H. Hult, A. Proutiere, S. Samama, A. Tarighati. A generative model of a limit order book using recurrent neural networks. *Quantitative Finance* 23(6):931–958, 2023.
- [14] A. Kirilenko, A. S. Kyle, M. Samadi, T. Tuzun. The Flash Crash: high-frequency trading in an electronic market. *Journal of Finance* 72(3):967–998, 2017.
- [15] A. S. Kyle. Continuous auctions and insider trading. *Econometrica* 53(6):1315–1335, 1985.
- [16] P. Nagy, S. Frey, K. Li, B. Sarkar, S. Vyetenko, S. Zohren, A. Calinescu, J. Foerster. LOB-Bench: benchmarking generative AI for finance — an application to limit order book data. arXiv:2502.09172, 2025.
- [17] A. A. Obizhaeva, J. Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets* 16(1):1–32, 2013.
- [18] T. Ogata. The geometry of diffusion: Fokker–Planck flow, Wasserstein gradient flows, and reverse-time SDEs. Companion survey, 2026.
- [19] T. Ogata. From score to flow map: equivalent representations of generative diffusion. Companion survey, 2026.
- [20] T. Ogata. Steering the Fokker–Planck equation: natural-gradient, optimal-transport, and score-based density control. Companion survey, 2026.
- [21] Z. Wang, C. Ventre. DiffVolume: diffusion models for volume generation in limit order books. arXiv:2508.08698, 2025.
- [22] Z. Zhang, S. Zohren, S. Roberts. DeepLOB: deep convolutional neural networks for limit order books. *IEEE Transactions on Signal Processing* 67(11):3001–3012, 2019.