

A Volatility-Based Single Parameter LGD Model

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SUMMARY: This paper proposes a new loss given default (LGD) model that links LGD to the default rate through a single parameter, asset volatility. The model is designed for wholesale credit risk management and can be applied to LGD forecasting, stress-testing, and integration into the Basel Advanced Internal Rating-Based (AIRB) framework. It also has potential applications in market risk management, particularly for deriving market-based LGD for regulatory credit valuation adjustment (CVA) under the Basel Advanced CVA capital framework.

METHOD: The model is based on a simplified Merton corporate debt model and uses a conditional lognormal distribution to derive LGD. The single parameter, asset volatility, is calibrated using data on probability of default (PD), expected loss (EL), and correlation factor (ρ). The model is compared to existing LGD models, such as those proposed by Frye and Tasche.

KEY FINDINGS:

- The proposed LGD Mapping Function (LMF) is a closed-form equation that simplifies the calibration process.
- The model is robust and can be extended to factor in debt seniorities.
- The asset volatility parameter has intuitive economic meaning and correlates well with the credit cycle.
- The model can be easily implemented in wholesale credit risk management and integrated into the Basel AIRB framework.

IMPLICATION FOR TRADING: This model provides a straightforward and efficient way to estimate LGD, which is crucial for risk management and regulatory compliance. Practitioners can use this model to improve their LGD forecasts and stress-testing processes, ensuring better risk assessment and management. The model's simplicity and robustness make it particularly useful for integrating into existing risk management systems.

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Abstract

This paper proposes an alternative recovery model (i.e. LGD Mapping Function) that links loss given default (LGD) to default rate through a single asset volatility parameter. The model is closed-form and easy to calibrate. It may be applied to wholesale credit risk management, such as LGD forecast, stress-testing and integration into Basel advanced internal rating-based (AIRB) framework. The model may also potentially be applied to market risk management, such as derivation of market-based LGD for the purpose of regulatory credit valuation adjustment (CVA) under Basel Advanced CVA capital framework.

JEL Classification: G10, G18, G21, G28

Keywords: Basel, CVA, default, LGD, volatility, CDS

Organization of the paper

Section 1 introduces the background and relevant research; Section 2 summarizes the contribution of the paper; Section 3 presents the proposed LGD Mapping Function (LMF); Section 4 discusses the calibration of the single parameter asset volatility; Section 5 discusses potential application to wholesale credit risk management; Section 6 discusses the potential application to counterparty CVA; Section 7 summarizes the conclusion and limitations. Appendix A summarizes the mathematical derivation of LMF, Appendix B list the key data used in the paper and Appendix C summarizes the derivation of the extended LMF to factor in debt seniorities.

Section 1 Introduction

Empirical data on default and recovery history showed a negative correlation between default rates and recovery rates as summarized by Altman, E.I., Brady, B., Resti, A. and Sironi, A. (2005)¹. It makes sense to consider the impact of default rate when modeling recovery or loss given default rate (i.e. LGD, which is equal to $1 - \text{recovery rate}$). Frye (2000)⁸ introduced an approach that modelled default and recovery on joint Vasicek distribution associated with the systematic risk factor. Pykhtin (2003)²¹ proposed a model that allows collateral value to follow lognormal distribution with links to idiosyncratic

borrower's risk and systematic risk through a separate recovery correlation factor. Tasche (2004)²⁴ proposed expanding the asymptotic single risk factor (ASRF) default risk model of Gordy (2003)¹¹ by incorporating LGD volatility and fitting loss over Beta-distribution. Hillebrand (2006)¹³ suggested an integrated model for the portfolio loss, contrary to the current Basel approach of modeling PD (probability of default) and LGD separately. Chava, Stefanescu & Turnbull (2006)⁴ used issuer-specific, time-dependent data to model default and recovery rates simultaneously. Miu and Ozdemir (2006)¹⁹ developed a downturn LGD model explicitly considering systematic and idiosyncratic PD-LGD correlations. Hamerle, Knapp and Wildenauer (2007)¹² introduced a model that split the systematic risk associated to default and LGD into an observable and an unobservable part. Jacobs (2011)¹⁴ developed an option-based model combining the effect of systematic risk and stochastic returns on defaulted debt to estimate ultimate recovery rate. Frye and Jacobs (2012)⁹ and Frye (2014)¹⁰ proposed a simple LGD function linking default rate to LGD by using a single parameter K that is calibrated over PD, EL and correlation factor ρ . Different from Gordy (2003)¹¹ and Finger (1999)⁷ who assumed that only default follows Vasicek distribution²⁵, Frye assumed that loss and default share a pair of comonotonic Vasicek distributions with the same risk quantile and correlation ρ to the systematic risk. The research mentioned above generally cover LGD modeling related to wholesale credit risk. The topic of deriving market-based LGD from credit default swap (CDS) spreads and other market-tradable instruments will be mentioned in Section 6.

Section 2 Contributions of this paper

The author in this paper proposed a closed-form LGD model (namely LGD Mapping Function or LMF) linking LGD to default rate through a single parameter, asset (return) volatility σ . The proposed model uses a similar framework as Yang (2013)²⁶ but is enhanced through further simplification. It bears some resemblance to the models proposed by Tasche²⁴ and Frye⁹. The key difference between Tasche's model and LMF is: Tasche factors LGD correlation effect into an expanded ASRF model and fits loss into Beta distribution whereas the proposed LMF integrates a simplified Merton corporate debt model¹⁸ into conditional lognormal distribution to derive LGD. Similar to Frye's LGD function, LMF is also a closed-form equation linking PD to LGD over a single parameter. The key difference is: Frye assumes loss and default follow comonotonic Vasicek distributions with the same correlation factor to the systematic risk captured in the "LGD risk index" parameter K ; the proposed LMF assumes asset value follows lognormal distribution and uses asset volatility σ as the single parameter linking PD to LGD. It does not assume that loss and default follow a comonotonic Vasicek distribution.

Contributions of this paper are as follows:

1. The parameter calibration and LGD calculation of the proposed LMF is simple and requires as few as two inputs;
2. LMF is robust and can be extended to factor in debt seniorities;
3. The asset volatility parameter in LMF has intuitive economic meaning and correlates well with the credit cycle. It is easy to implement LMF in wholesale credit risk management, such as LGD forecast or stress-testing. LMF may also be easily integrated into the current Basel advanced internal rating-based (AIRB) framework;

4. LMF may also be applied to market risk management. The asset volatility parameter can be derived from equity dynamics through well-known methodology and therefore marked-to-market on daily basis if required. It can also be easily extended to horizons other than one year. LMF may be used to derive market-based LGD, one of the key inputs into the regulatory credit valuation adjustment (CVA) measurement under the current Basel Advanced CVA capital framework.

Section 3 Derivation of LGD Mapping Function

Assuming that a firm's asset value follows lognormal distribution and a one-year horizon, one can easily derive the LGD Mapping Function from a simplified Merton model¹⁸ and the property of lognormal distribution.

$$PD = \Phi\left(\frac{\ln D - \ln V_0 - \left(r - \frac{\sigma^2}{2}\right)}{\sigma}\right) \quad (1)$$

$$V_d = \frac{1}{PD} \int_0^D v \cdot \frac{1}{v\sigma\sqrt{2\pi}} e^{-\frac{(\ln v - \mu)^2}{2\sigma^2}} dv \quad (2)$$

Combining (1) and (2), one has the following:

$$LGD = 1 - e^{-\sigma\Phi^{-1}(PD) + \frac{\sigma^2}{2} \frac{\Phi[\Phi^{-1}(PD) - \sigma]}{PD}} \quad (3)$$

Note: V_0 = the asset value at the time of estimation, D = constant level of debt over one year horizon, σ = annual asset return volatility, Φ = standard normal cumulative distribution function, PD = probability of default, r = expected asset return, v = log-normally distributed stochastic variable that represents the asset value at the default horizon, V_d = expected asset value given default.

For detailed derivation please see Appendix A.

Equation 3 is the proposed LGD Mapping Function (LMF) with asset volatility σ as the single parameter. Instead of estimating the relative change of LGD against the change of PD as in Yang (2013)²⁶, LMF directly links LGD to its corresponding PD.

The asset volatility parameter σ requires only two inputs, PD and LGD, for calibration. To calculate LGD from default rate, LMF requires only one input, i.e. the default rate for the same time period as the LGD to be calculated. Since LMF takes PD as an exogenous input, one may have the flexibility of using other models or expert judgement to stress PD and then feed the stressed PD into LMF to obtain the stressed LGD. The separation of PD and LGD stressing is consistent with the current Basel AIRB approach^{2 & 3}. In addition to stressing LGD through stressed PD, users of LMF may also stress LGD by stressing the asset volatility parameter σ .

Table 3.1 and Figure 3.1 show the relationship among the three variables in LMF: given the same asset volatility, higher PD leads to higher LGD; given the same PD, higher asset volatility leads to higher LGD.

Table 3.1 Relationship among PD, σ and LGD

	PD	0.10%	0.50%	1.00%	5.00%	10.00%
σ	0.50	12.24%	13.77%	14.64%	17.56%	19.54%
	1.00	21.91%	24.35%	25.72%	30.20%	33.14%
	2.00	36.15%	39.46%	41.26%	46.95%	50.51%
	4.00	53.40%	56.98%	58.85%	64.51%	67.85%
	8.00	69.88%	72.89%	74.43%	78.84%	81.30%

Note: the highlighted area is the LGD corresponding to the PD and σ .

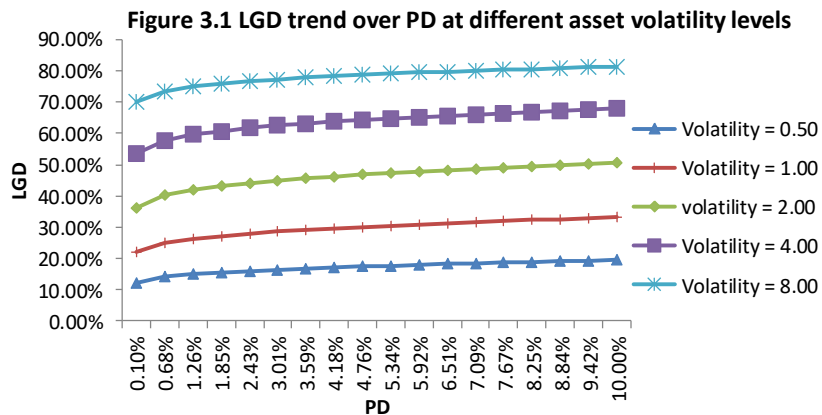
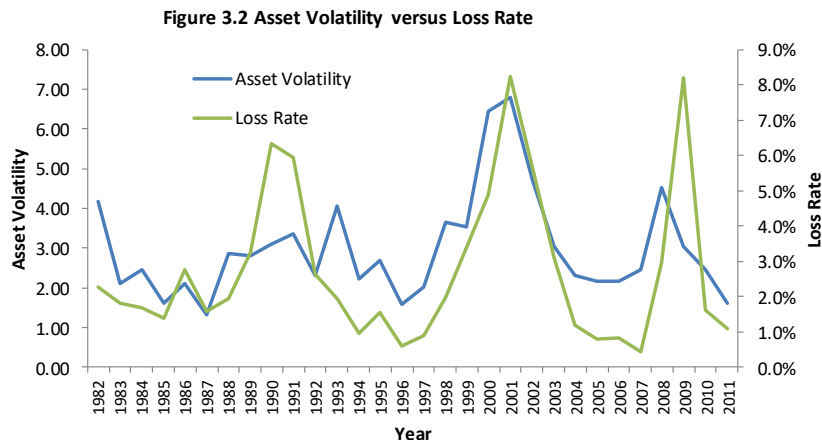


Figure 3.2 shows that the annual asset volatilities implied by LMF are positively correlated to the annual loss rates from 1982 to 2011, i.e. the credit cycles. The data is from Moody's (2012) annual default and recovery study²⁰. The annual asset volatilities are back-solved over LMF from the default rates for speculative grade corporate debt and the average recovery rates for senior unsecured debt. Appendix B shows the complete data time series.



The asset volatilities implied by LMF seem to be high but are actually intuitive. The volatility is defined as the standard deviation of logarithm asset value. When large LGD occurs, the changes between the ending log asset value and the initial log value must be large, leading to a larger standard deviation, i.e. volatility.

Section 4 Calibration of Asset Volatility

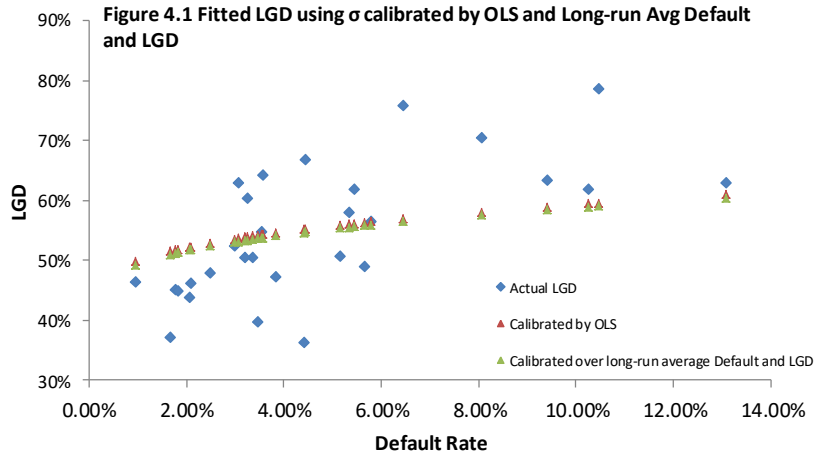
Two approaches for calibrating asset volatility σ are presented in this section: (a) using ordinary least square (OLS) to find the σ that minimizes the root mean square error (RMSE) for estimated LGDs against realized LGDs over the historic time series; (b) directly back-solve σ from just one pair of data points, the estimated long-run average PD and LGD. The same Moody's (2012) time series as in Section 3 is used.

Table 4.1 lists the calibrated σ under these two approaches and shows that RMSEs are very close. Figure 4.1 shows that the fitted LGD time series are almost identical.

Note: the long-run average PD is 4.63% and LGD is 54.89%, based on the arithmetic average of the same Moody's (2012) 30-year time series. When historical data are available, the first approach is favoured as the mean error is much smaller than the second approach.

Table 4.1 Calibration of Asset Volatility σ

	(a)	(b)
Parameter Calibration	Calibrated over historical default and LGD time series by OLS	Calibrated from long-run average default rate and LGD
Asset Volatility σ	2.8119	2.7548
RMSE	9.17%	9.19%
Mean Error	-0.02%	-0.55%
Variance	0.87%	0.87%



When historical data is very scarce, one may still calibrate σ using as few as one pair of PD and LGD data, such as the default rate and LGD of the previous quarter or year. Using only the previous year's default rate DR_{t-1} and LGD_{t-1} from the same Moody's historical data, the author back-solved volatility σ_{t-1} from LMF:

$$LGD_{t-1} = 1 - e^{-\sigma_{t-1}\Phi^{-1}(DR_{t-1}) + \frac{\sigma_{t-1}^2}{2}} \frac{\Phi[\Phi^{-1}(DR_{t-1}) - \sigma_{t-1}]}{DR_{t-1}}$$

The LGD for the current year LGD_t is then estimated by plugging the default rate of the current year DR_t and previous year's σ_{t-1} into LMF:

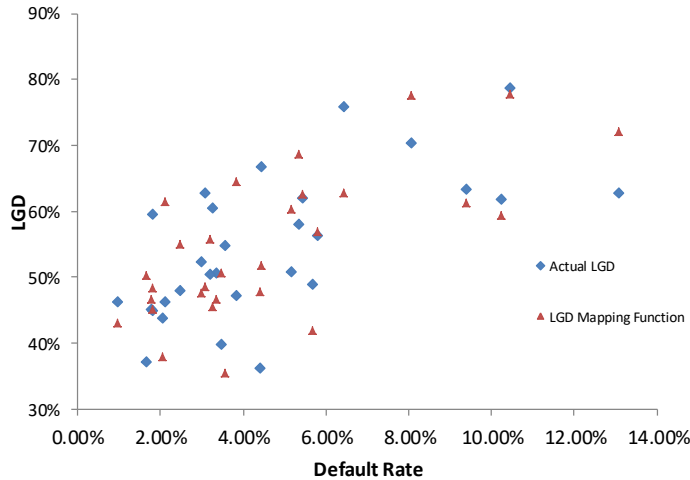
$$LGD_t = 1 - e^{-\sigma_{t-1}\Phi^{-1}(DR_t) + \frac{\sigma_{t-1}^2}{2}} \frac{\Phi[\Phi^{-1}(DR_t) - \sigma_{t-1}]}{DR_t}$$

Table 4.2 and Figure 4.2 show the LGD prediction errors. The result is not much worse than the previous calibration results in Table 4.1.

Table 4.2 LGD Prediction Errors for 1983-2011

LGD Prediction for rolling years 1983-2011	Calibrated over Previous Year Data Only (1982 up)
Min σ	1.33
Max σ	6.79
Mean σ	2.99
RMSE	9.89%
Mean Error	0.05%
Variance	1.01%

Figure 4.2 LGD Prediction Errors with σ calibrated over previous year's data only



Section 5 Application of LGD Mapping Function to Wholesale Credit Risk

1. Forecasting LGD with default rate known

The resolution of a default is usually determined long after the default is recognized. Factors such as resolution period, workout procedures and discount rate for recovered cash flows often make the assessment of LGD for defaulted loans very difficult. For a given period such as a quarter or a year, one may estimate the realized default rate sooner and with greater accuracy than one can estimate LGD of the defaulted loans. LMF may provide a convenient way to calculate an expected LGD based on estimated or realized default rate. The LGDs calculated from LMF can be used as a benchmark or challenge metric to assess the reasonableness and conservatism of banks' internal LGD estimates.

Given pre-calibrated σ and the estimated default rate DR_1 , LMF calculate the expected LGD_1 for the given quarter or year as follows:

$$LGD_1 = 1 - e^{-\sigma \Phi^{-1}(DR_1) + \frac{\sigma^2}{2} \frac{\Phi[\Phi^{-1}(DR_1) - \sigma]}{DR_1}}$$

2. Stress testing LGD

One may input a stressed default rate into LMF to calculate a stressed LGD. Since the default rate input in LMF is not bound by loss distribution, the stressed default rate can be based on various methods: historical downturn, expert judgement, or some PD stressing models.

For example, banks may use their internal downturn default rate and LGD to calibrate a "downturn σ " to stress test the LGD. If Year 2009 is defined as a downturn, the downturn default rate is 13.08% and the downturn LGD is 62.89% based on the same Moody's data. If the bank predicts the next downturn would see a default rate of 15%, it feeds 15% into LMF to get a downturn LGD. The question is what σ to be used? Table 5.2.1 shows

the stress testing results using three different σ calibrated from lowest to highest: using long-run average PD and LGD ($\sigma=2.7548$), using downturn PD and LGD ($\sigma=3.0359$) and using the historical highest LGD 78.76% and its corresponding PD 10.46% ($\sigma=6.7859$).

Table 5.2.1 Stress testing LGD using LMF

	Downturn PD	Downturn LGD	σ	rho	CL	Stressed PD	Stressed LGD	Relative Change %
1	13.08%	62.89%	2.7548			15.00%	61.40%	-2.37%
2	13.08%	62.89%	3.0359			15.00%	63.81%	1.46%
3	13.08%	62.89%	6.7859			15.00%	80.44%	27.90%

Note: Neither rho nor CL is needed for LMF to calculate LGD, where CL is the confidence level for the stressing of the systematic risk factor under ASRF model. Relative change% is the relative change of the stressed LGD over the historical downturn LGD.

Using the downturn σ corresponding to the historical downturn PD and LGD (#2) ensures that the stressed LGD will always be higher than the downturn LGD as long as the stressed PD is higher than the downturn PD. However, to produce stronger stressing, one may choose #3 or even higher σ based on expert judgement.

In addition to using historical data, one may use a PD model to stress PD first and then input it into LMF to stress-test LGD.

Table 5.2.2 shows the stress testing of LGD where LMF takes stressed PD produced by ASRF model (Gordy 2003)¹¹ as the input. The long-run average PD and LGD are used as the baseline PD and LGD.

Table 5.2.2 Stressing LGD with ASRF-stressed PD

Long-run PD	Long-run LGD	σ	rho	CL	Stressed PD	Stressed LGD	Relative Change %
4.63%	54.89%	2.7548	0.00%	90.00%	10.76%	59.24%	7.92%
4.63%	54.89%	2.7548	20.00%	95.00%	14.50%	61.17%	11.44%
4.63%	54.89%	2.7548	20.00%	99.00%	23.66%	65.00%	18.42%
4.63%	54.89%	2.7548	20.00%	99.90%	36.87%	69.62%	26.84%

Note: σ is calibrated on long-run average PD and LGD. Rho and CL are required by ASRF, not by LMF itself.

3. Integration into Basel Advanced Internal Rating based Approach (AIRB) framework.

The current Basel AIRB framework explicitly stresses PD to 99.9% confidence level using ASRF model while leaving LGD as an exogenous input that should reflect the “downturn” credit cycle (i.e. stressed LGD):

$$Capital(K) = LGD_{downturn} \times \left(\phi \left\{ \frac{\phi^{-1}(PD) + \sqrt{\rho} \phi^{-1}(CL)}{\sqrt{1-\rho}} \right\} - PD \right)$$

Let SLGD be the stressed LGD and SPD be the stressed PD. SPD is calculated by

ASRF model as:
$$SPD = \phi \left\{ \frac{\phi^{-1}(PD) + \sqrt{\rho} \phi^{-1}(CL)}{\sqrt{(1-\rho)}} \right\}$$

The capital formula can be rewritten as: $Capital(K) = SLGD \times (SPD - PD)$

The current Basel framework does not necessarily require SPD and SLGD to be mathematically linked, such as following a joint distribution.

Since LMF uses PD as an exogenous input to calculate the corresponding LGD, it can be easily integrated into the Basel AIRB formula by replacing SLGD with LMF-stressed LGD:

$$Capital(K) = SLGD_{LMF} \times (SPD - PD), \text{ where } SLGD_{LMF} = 1 - e^{-\sigma \Phi^{-1}(SPD) + \frac{\sigma^2}{2} \frac{\Phi[\Phi^{-1}(SPD) - \sigma]}{SPD}} \text{ and}$$

$$SPD = \phi \left\{ \frac{\phi^{-1}(PD) + \sqrt{\rho} \phi^{-1}(CL)}{\sqrt{(1-\rho)}} \right\}$$

The integrated capital formula will calculate both a stressed PD to a specific confidence level (such as 99.9%) and a stressed LGD that is mathematically associated with the stressed PD.

The rationale for such integration is as follows:

It is possible to link SPD under ASRF model with SLGD under LMF. The ASRF model stresses PD via a single systematic risk factor $\phi^{-1}(CL)$ as:

$$SPD_{ASRF} = \phi \left\{ \frac{\phi^{-1}(PD) + \sqrt{\rho} \phi^{-1}(CL)}{\sqrt{(1-\rho)}} \right\}$$

SPD_{ASRF} can then be regarded as the “new” expected PD in the stressed systematic environment.

Under Merton framework, PD may be driven by expected return of asset “r”, if all other terms are held constant. Given that SPD_{ASRF} is the new expected PD, one can stress PD under Merton framework to the same level of SPD_{ASRF} by stressing the expected return of asset “r”:

$$SPD_{Merton} = \Phi \left[\frac{LnD - LnV_0 - \left(r_s - \frac{\sigma^2}{2} \right)}{\sigma} \right]$$

Note: r_s is the stressed asset return and $SPD_{Merton} = SPD_{ASRF}$

Intuitively, this can be seen as shifting the asset value distribution to the left. The corresponding SLGD under this shifted lognormal distribution can also be described by LMF as a function of SPD:

$$SLGD_{LMF} = 1 - e^{-\sigma\Phi^{-1}(SPD) + \frac{\sigma^2}{2}} \frac{\Phi[\Phi^{-1}(SPD) - \sigma]}{SPD}$$

The capital is then given by: $Capital(K) = SLGD_{LMF} \times (SPD - PD)$

For detailed explanation on combining ASRF and Merton framework, please refer to Yang (2013)²⁶.

Table 5.3.1 shows the capital factor K at 99.9% confidence level. Capital K is impacted by both rho and σ . Correlation rho drives stressed PD, which in turn drives stressed LGD. Volatility σ directly drives stressed LGD. Similar to rho, σ can also be set by regulators.

Table 5.3.1 Capital K by integrating LMF into current Basel AIRB framework

Long-run PD	Long-run LGD	σ	rho	CL	Stressed PD	Stressed LGD	Capital K
4.63%	54.89%	2.7548	12.00%	99.90%	25.73%	65.77%	13.88%
4.63%	54.89%	2.7548	24.00%	99.90%	42.36%	71.42%	26.95%
4.63%	54.89%	2.7548	18.00%	99.90%	34.11%	68.70%	20.25%
4.63%	54.89%	4.0995	18.00%	99.90%	34.11%	77.18%	22.75%

Note: $\sigma = 2.7548$ is based on long-run PD and LGD while $\sigma = 4.0995$ is based on 84% percentile highest σ calculated from the same Moody's historical data.

Table 5.3.2 Stressed PD and LGD with Capital K across different ratings

Rating	PD	LGD	σ	rho	CL	Stressed PD	Stressed LGD	Capital K
Aaa	0.01%	60.00%	6.0771	18.00%	99.90%	0.39%	66.45%	0.25%
Aa	0.02%	62.76%	6.4791	18.00%	99.90%	0.77%	69.47%	0.52%
A	0.07%	68.23%	7.6264	18.00%	99.90%	1.86%	75.05%	1.34%
Baa	0.20%	58.61%	4.6506	18.00%	99.90%	4.19%	67.30%	2.69%
Ba	1.15%	52.89%	3.1079	18.00%	99.90%	14.39%	64.09%	8.48%
B	4.21%	62.12%	3.7340	18.00%	99.90%	32.30%	74.80%	21.01%
Caa	14.36%	64.28%	3.1335	18.00%	99.90%	60.75%	79.94%	37.08%

Note: the asset volatility σ is calibrated to the corresponding PD and LGD for each rating. The investment grade has higher asset volatilities than non-investment grade due to effects such as fallen angels and low default rates. The PD and LGD are based on Moody's (2012)²⁰ except that PD and LGD for Aaa rating are assumed to be 0.01% and 60% respectively.

4. Application of LMF to debt of different seniorities

a) Direct application of LMF to debt of different seniorities

LMF as presented in Equation 3 is essentially a "single-tranche" LMF that calculates LGD of total debt of a firm. It does not differentiate debt seniorities. Having said so, one may still use LMF to assess debt of different seniorities. Given a firm's PD and the LGDs for debt of different seniorities, one may use LMF to back-solve the "implied volatility σ " corresponding to each seniority.

Table 5.4.1 Implied asset volatilities for different seniorities

Lien Position	PD	Recovery	LGD	Implied σ
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Senior Secured Bonds	4.63%	63.70%	36.30%	1.32
Senior Unsecured Bonds	4.63%	48.50%	51.50%	2.42
Subordinated Bonds	4.63%	28.70%	71.30%	5.48

Note: PD is assumed to be the same long-run average PD and the average recovery rate is based on ultimate recovery for average corporate debt 1987-2011 in Moody's (2012)²⁰. The debt structure information (such as ratio of senior to subordinate debt) is not known.

Table 5.4.1 shows that LMF-implied σ displays a reasonable upward trend as the seniority of the debt goes down. The implied σ of subordinated debt is higher than that of senior unsecured debt, reflecting the higher risk of the residual assets supporting the subordinate debts. By the same token, implied σ for secured debt should be smaller, reflecting lower risk as secured assets (i.e. collateral) stand to back the debt. The implied asset volatilities derived this way may be fed into LMF to estimate LGDs for senior and subordinated debt of individual firms with firm-specific PDs.

b) Extension of LMF to factor in the debt structure

When the debt structure is known, it is also possible to extend LMF to include the debt structure into the formula (i.e. Extended LMF or ELMF). ELMF estimates LGDs for debts of different seniorities as a function of the firm's PD, the overall asset volatility σ and the debt structure.

ELMF for senior debt D_1 :

$$LGD_1 = \frac{1}{P \bullet d_1} \left[d_1 \Phi \left(\Phi^{-1}(P) + \frac{\ln(d_1)}{\sigma} \right) - e^{-\sigma \Phi^{-1}(P) + \frac{\sigma^2}{2}} \Phi \left(\Phi^{-1}(P) - \sigma + \frac{\ln(d_1)}{\sigma} \right) \right] \quad (4)$$

ELMF for subordinated debt D_2 :

$$LGD_2 = 1 - \frac{1}{P \bullet d_2} \left[e^{-\sigma \Phi^{-1}(P) + \frac{\sigma^2}{2}} \Phi \left(\Phi^{-1}(P) - \sigma \right) - (1 - LGD_1) \bullet P \bullet d_1 \right] \quad (5)$$

Note: $D = D_1 + D_2$, $d_1 = D_1 / D$, $d_2 = D_2 / D$, P is firm's PD and σ is the volatility of the firm's asset.

For more detailed derivation of ELMF and its application, please refer to Appendix C.

Section 6 Application of LGD Mapping Function to Market Risk

1. The issue about market-based LGD

The current Basel counterparty credit risk (CCR) measures not only the default risk of a derivative counterparty but also the deterioration risk of its credit worthiness short of default. This latter risk is measured via credit valuation adjustment (CVA) as follows:

$$CVA = LGD_{MKT} \cdot \sum_{i=1}^T \text{Max} \left[0; e^{-\frac{s_{i-1} \cdot t_{i-1}}{LGD_{MKT}}} - e^{-\frac{s_i \cdot t_i}{LGD_{MKT}}} \right] \cdot \left[\frac{EE_{i-1} \cdot D_{i-1} + EE_i \cdot D_i}{2} \right]$$

T is the final maturity of the evaluated instrument, LGD_{MKT} is the market-based LGD, s_i is the credit spread for tenor t_i , and EE_i and D_i are the expected exposure and discount factor for the i^{th} revaluation time point respectively. The CVA formula essentially calculates the market price for credit risk as the sum of discounted expected credit loss over each revaluation interval ($t_i - t_{i-1}$). For detailed explanation of the regulatory CVA formula, please refer to Basel (2011)³.

Since CVA calculates the market price for credit risk, Basel requires that the key inputs LGD_{MKT} , the credit spread for each tenor S_t and the expected exposures EE_t should be market-based. Basel specifically emphasizes that LGD_{MKT} should be based on the spread of a market instrument of the counterparty (or where a counterparty instrument is not available, based on the proxy spread that is appropriate based on the rating, industry and region of the counterparty). This LGD_{MKT} is different from the LGD that is determined for the AIRB and CCR default risk charge, as this LGD_{MKT} is a market assessment rather than an internal estimate.

Both spread S_t and expected exposures can be directly observed from the market, i.e. S_t by credit default swap (CDS) spreads and exposures by mark-to-market valuation. The only input that is not directly observable in the market is LGD_{MKT} . Although CDS spread S_t is a market-based measure of expected credit loss combining the effect of default, or more precisely default intensity, and LGD, it does not inform how these two components should be split. The current market practice for pricing CDS or CVA is often assuming a constant LGD_{MKT} based on the market convention unless one can directly observe the LGD_{MKT} from a defaulted instrument, which is rare.

The literature on how to derive market-based LGD is relatively new and small. To name a few such as Zhang (2006)²⁷ who assumed that both default intensity and the recovery rate are driven by the same factor that governs the risk-free rate term structure. Das and Hanouna (2009)⁵ developed a parametric solution to link PD with stochastic LGD and derived market implied LGD from CDS term structure and equity price. Le (2007)¹⁶ inferred implied recovery rates in a two-step procedure. Seidler, J. and Jakubik, P. (2009)²³ proposed a diffusion LGD model based on equity market information and other structural company parameters. Doshi (2011)⁶ used a stochastic recovery model fitting CDS data. Jaskowski and McAleer (2012)¹⁵ proposed a parsimonious reduced-form continuous time model that estimates recovery and PD from the term structure of CDS spreads by estimating the parameters and latent factors over a Bayesian algorithm.

2. The potential solution through LMF

This section proposes a potential solution using LMF to derive the market-based LGD from CDS spreads. The idea bears some resemblance to Das and Hanouna (2009)⁵, Le (2007)¹⁶ and Jaskowski and McAleer (2012)¹⁵ in terms of using equity dynamics and CDS spreads to derive PD and LGD term structures. Although it is less sophisticated and styled than these models (for example, the method is static instead of a jump-compensated stochastic process), it could still serve as a “short-cut” way to estimate the

market-based LGD for the Basel regulatory CVA measurement. The key point is: LMF is used here as the function linking PD and LGD. It has only one parameter σ to calibrate and can estimate PD and LGD term structures simultaneously from a given CDS spread term structure.

Simplifying notations, the CVA formula implies $s \bullet t = -LGD \bullet \ln(1 - CPD)$, where s is the CDS spread and CPD is the cumulative PD over the tenor t . If one replaces LGD with LMF(CPD, σ), where LMF takes CPD as the input, one obtains:

$$s_i \bullet t_i = -LMF(CPD_i, \sigma_i) \bullet \ln(1 - CPD_i)$$

Since CDS spreads are observable in the market, one may back-solve for CPD and LGD over LMF. *(Note that CPD in LMF is not estimated directly from the structural Merton model, but is estimated from CDS spreads as an exogenous input. The default intensity α , CPD and LGD have the following relationship: $s \bullet t = LGD \bullet \alpha \bullet t$ and $e^{-\alpha \bullet t} = 1 - CPD$)*

Using LMF as the LGD function here has the following advantages:

- The asset volatility parameter σ can be derived from equity price and equity volatility through Ito's lemma as demonstrated by Marcus, AJ and Shaked, I. (1984)¹⁷. It is market-based and can be marked-to-market on a daily basis;
- One can easily change the time horizon of σ to t-period horizons by $\sigma_t = \sigma \sqrt{t}$. There is no need to recalibrate the parameter using t-period's PD and LGD.

3. Examples of using LMF to derive market-based LGD term structure

Assuming the equity market-derived asset volatility is 20% for XYZ counterparty, fitting LMF into the CDS spreads one can obtain the term structures of the market-based CPD and LGD.

Table 6.3.2 shows the market-based LGDs derived from CDS spreads and market-based σ versus LGDs derived from CDS spreads but with σ from AIRB calibration. The market-based risk-neutral LGDs are smaller than the AIRB-based LGDs while the market-based PDs are larger than the AIRB-based PDs. This is consistent with observations from many of the previous literature.

Table 6.3.1 Market-based LGD versus LGD based on AIRB σ

CDS Spread	Tenor (years)	Market-based σ	CEL	CPD	LGD
1.00%	1	0.20	0.942%	10.68%	9%
1.17%	2	0.28	2.131%	16.19%	13%
1.33%	3	0.35	3.548%	21.18%	17%
1.50%	4	0.40	5.167%	25.91%	20%
1.67%	5	0.45	6.966%	30.44%	23%

CDS Spread	Tenor	AIRB-based σ	CEL	CPD	LGD
1.00%	1	2.75	0.990%	1.92%	51%
1.17%	2	3.90	2.290%	3.66%	63%
1.33%	3	4.77	3.886%	5.63%	69%
1.50%	4	5.51	5.762%	7.83%	74%
1.67%	5	6.16	7.898%	10.27%	77%

Note: For each tenor, CPD is the cumulative default rate, LGD is the average LGD and CEL is the cumulative expected loss.

4. Proposing a modified CVA formula to integrate the market-based LGD term structure

To incorporate the market-based LGD term structure into the regulatory CVA calculation, the current CVA formula must be modified as such:

$$CVA = \sum_{i=1}^T \text{Max} \left[0; LGD_{MKT_i} \cdot \left(1 - e^{-\frac{s_i \cdot t_i}{LGD_{MKT_i}}} \right) - LGD_{MKT_{i-1}} \cdot \left(1 - e^{-\frac{s_{i-1} \cdot t_{i-1}}{LGD_{MKT_{i-1}}}} \right) \right] \cdot \left[\frac{EE_{i-1} \cdot D_{i-1} + EE_i \cdot D_i}{2} \right]$$

Table 6.4.1 Demo of the modified CVA calculation over shocks (flat discount rate 5%)

Day 1: The equity-derived asset volatility was assumed to be 20%.

CDS Spread	Tenor	σ	CEL	CPD	LGD	EE	DF	Expected Loss for each interval
1.00%	1	0.20	0.942%	10.68%	9%	10%	0.95	0.04%
1.17%	2	0.28	2.131%	16.19%	13%	20%	0.91	0.16%
1.33%	3	0.35	3.548%	21.18%	17%	30%	0.86	0.31%
1.50%	4	0.40	5.167%	25.91%	20%	15%	0.82	0.31%
1.67%	5	0.45	6.966%	30.44%	23%	0%	0.78	0.11%

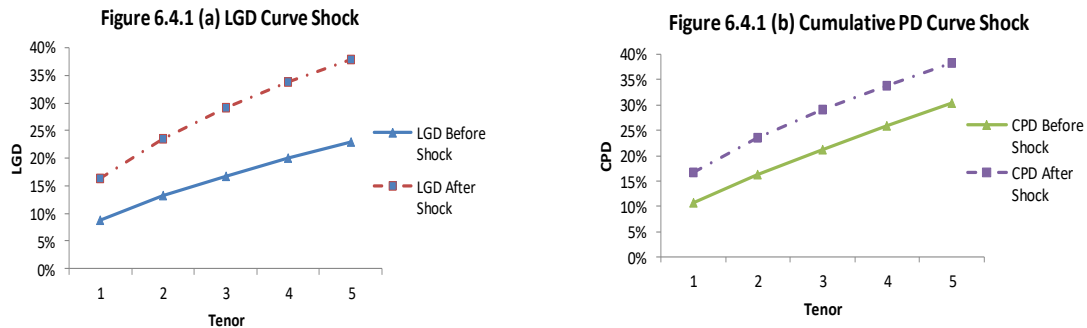
CVA 0.94%

Day 2: A shock occurred to CDS and equity market on XYZ. The CDS spreads jumped by 200 bps for all tenors and the equity-derived asset volatility rose to 36%.

CDS Spread	Tenor	σ	CEL	CPD	LGD	EE	DF	Expected Loss for each interval
3.00%	1	0.36	2.741%	16.77%	16%	10%	0.95	0.13%
3.17%	2	0.51	5.554%	23.54%	24%	20%	0.91	0.39%
3.33%	3	0.62	8.465%	29.04%	29%	30%	0.86	0.64%
3.50%	4	0.72	11.463%	33.91%	34%	15%	0.82	0.57%
3.67%	5	0.80	14.535%	38.36%	38%	0%	0.78	0.19%

CVA 1.92%

CVA is increased from 94 bps to 192 bps. Figure 6.4.1 (a) and (b) show the impact of the shock on LGD and CPD term curves. Under the modified CVA formula, both shocks on the CDS spread and shocks on the equity market will impact CVA.



Section 7 Conclusion and Limitations

This paper introduced a simple LGD Mapping Function that links LGD with default rate through a single asset volatility parameter. The model is easy to calibrate and may be used to forecast or stress-test LGD, to integrate stressed LGD into Basel AIRB capital framework and to derive the market-based LGD term structures for Basel Advanced CVA measurement.

Due to limitations of the data and the depth of the author's research, many topics such as empirical validation of LMF, comparison with other LGD models and potential applications to other areas are not discussed and may be open for further studies. The author presents this paper with the intention to prompt more fruitful discussions and research in the future.

Disclaimer

The views expressed in this paper are the author's own and do not necessarily reflect the view of Office of the Superintendent of Financial Institutions (OSFI).

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Appendix A Derivation of Equation 3

The expected asset value given default under lognormal distribution is defined as:

$$V_d = \frac{1}{PD} \int_0^D v \cdot \frac{1}{v\sigma\sqrt{2\pi}} e^{-\frac{(\ln v - \mu)^2}{2\sigma^2}} dv$$

One can easily obtain the following:

$$V_d = \frac{1}{PD} e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\ln D - \mu - \sigma^2}{\sigma}\right)$$

Φ is the cumulative standard normal distribution function.

According to Merton corporate debt model¹⁸:

$$PD = \Phi \left[\frac{\ln D - \ln V_0 - \left(r - \frac{\sigma^2}{2} \right)}{\sigma} \right] \text{ and } \mu = \ln(V_0) + r - \frac{\sigma^2}{2}, \text{ assuming } t=1.$$

For detailed derivation, please refer to Appendix A of Yang (2013)²⁶ or Appendix 12A of Resti, A. and Sironi, A. (2007)²².

Defining recovery rate R as V_d/D , one then has:

$$R = \frac{1}{PD \bullet D} e^{\mu + \frac{\sigma^2}{2}} \Phi \left(\frac{\ln D - \mu - \sigma^2}{\sigma} \right) \quad (\text{A.1})$$

$$\text{Given } \mu = \ln D - \sigma \Phi^{-1}(PD) \quad (\text{A.2})$$

Substituting A.2 into A.1 and letting $LGD = 1 - R$, one then obtains Equation 3:

$$LGD = 1 - e^{-\sigma \Phi^{-1}(PD) + \frac{\sigma^2}{2}} \frac{\Phi[\Phi^{-1}(PD) - \sigma]}{PD}$$

Note: D is cancelled out after the substitution. This proves that normalizing D to 1 does not lose the generality of Equation 3. Some researchers such as Jacobs, M. (2011)¹⁴ also normalized debt to 1 in their model.

Appendix B Dataset based on Moody's²⁰

Year	PD Speculative Rate Corporate	Recovery Sr. Unsecured Bond	LGD = 1- Recovery
1982	3.57%	35.79%	64.21%
1983	3.83%	52.72%	47.28%
1984	3.35%	49.41%	50.59%
1985	3.47%	60.16%	39.84%
1986	5.67%	51.10%	48.90%
1987	4.41%	63.75%	36.25%
1988	3.56%	45.24%	54.76%
1989	5.78%	43.57%	56.43%
1990	10.24%	38.16%	61.84%
1991	9.40%	36.66%	63.34%
1992	5.16%	49.19%	50.81%
1993	3.08%	37.13%	62.87%
1994	2.10%	53.73%	46.27%
1995	2.99%	47.60%	52.40%
1996	1.66%	62.75%	37.25%
1997	2.06%	56.10%	43.90%
1998	3.25%	39.54%	60.46%
1999	5.45%	38.02%	61.98%
2000	6.44%	24.16%	75.84%
2001	10.46%	21.24%	78.76%
2002	8.06%	29.53%	70.47%
2003	5.34%	41.87%	58.13%
2004	2.48%	52.09%	47.91%
2005	1.77%	54.88%	45.12%
2006	1.82%	55.02%	44.98%
2007	0.97%	53.66%	46.35%
2008	4.44%	33.15%	66.85%
2009	13.08%	37.11%	62.89%
2010	3.20%	49.55%	50.45%
2011	1.82%	40.43%	59.57%

Appendix C Extending LMF to Factor in Debt Seniorities

Let D_1 and D_2 be the senior and subordinate debt of the same firm so that total debt $D = D_1 + D_2$. Let $d_1 = D_1 / D$, $d_2 = D_2 / D$ and P be the PD of the firm. Let $f(v, \mu, \sigma)$ be the lognormal density function with v , μ and σ defined the same way as in Appendix A. If R_1 and R_2 stand for the expected recovery rates for senior and subordinated debt respectively, one may establish the following relationship:

$$R_1 = \frac{1}{P \bullet D_1} \int_0^D \min(v, D_1) \bullet f(v, \mu, \sigma) dv$$

$$R_2 = \frac{1}{P \bullet D_2} \int_0^D \max(0, v - D_1) \bullet f(v, \mu, \sigma) dv$$

Expressing R_1 into LGD_1 and simplifying the integral into:

$$LGD_1 = 1 - \frac{1}{P \bullet d_1} \left[e^{-\sigma \Phi^{-1}(P) + \frac{\sigma^2}{2}} \Phi\left(\Phi^{-1}(P) - \sigma + \frac{\ln(d_1)}{\sigma}\right) + P \bullet d_1 - d_1 \Phi\left(\Phi^{-1}(P) + \frac{\ln(d_1)}{\sigma}\right) \right]$$

(Note: D_1 and D_2 become d_1 and d_2 respectively since D is normalized to 1.)

One then obtains the following extended LMF (i.e. ELMF) for senior debt:

$$LGD_1 = \frac{1}{P \bullet d_1} \left[d_1 \Phi\left(\Phi^{-1}(P) + \frac{\ln(d_1)}{\sigma}\right) - e^{-\sigma \Phi^{-1}(P) + \frac{\sigma^2}{2}} \Phi\left(\Phi^{-1}(P) - \sigma + \frac{\ln(d_1)}{\sigma}\right) \right] \quad (C.1)$$

Expressing R_2 into LGD_2 and simplifying the integral, one obtains the ELMF for the subordinated debt of the same firm:

$$LGD_2 = 1 - \frac{1}{P \bullet d_2} \left[e^{-\sigma \Phi^{-1}(P) + \frac{\sigma^2}{2}} \Phi\left(\Phi^{-1}(P) - \sigma\right) - (1 - LGD_1) \bullet P \bullet d_1 \right] \quad (C.2)$$

Assuming that PD of the firm is 4.63%, LGD for the senior debt is 51.50%, the ratio of the senior debt to total debt is 60% and the ratio of the subordinated debt is 40%, one can use ELMF to estimate the LGD for the subordinated debt.

Table C.1 LGD for senior and subordinated debt

Seniority	Ratio	PD	Recovery	LGD
Senior Debt	60%	4.63%	48.50%	51.50%
Subordinated Debt	40%	4.63%	10.31%	89.69%
Total Debt	100%	4.63%	33.22%	66.78%

The overall implied volatility σ by ELMF is 4.4680. One can test the validity of this σ by substituting it back into the original LMF (Equation 3) with PD of 4.63%. The derived LGD will be 66.78%, exactly equal to the weighted average LGD of the overall debt. (Note: it is different from Table 5.4.1 where no debt structure information is given and the LGDs are based on average recovery over many firms).

Alternatively, given the firm's PD, debt structure and asset σ , one can also use ELMF to calculate LGDs for both senior and subordinated debt of the same firm. ELMF may also be applied to securitization structures for the estimation of tranche-level losses.