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POVZETEK: Raziskava opisuje vpliv temne trga na cene in raziskovanje cene v teoriji trgovine z opcijskim modelom. Trgovci, ki uporabljajo temne trge, imajo obveznost izvajanja z določenim meškom, ki je povezan s asimetričnimi stroški, če ne dosežejo ali presežejo cilj. Temni trgi zmanjšajo cene in raziskovanje cene v primeru pozitivnih rizikov izvajanja, in to zmanjšanje postane neskončno, če se riziko izvajanja zmanjša. Če je presežen strošek dovolj manjši kot strošek nedospeka in vrednost privatne informacije, trgovci presežejo predvzemno v temne trge, ki je to v linearnem izjave neskončno.

METODA: Raziskava uporablja opcijski model, ki vključuje temen tržnem trgu in temen trgu, kjer je opcijska cena temne trga lognormalno porazdeljena. To omogoča reševanje izjave v zatvoreni obliki. Trgovci, ki uporabljajo temne trge, imajo obveznost izvajanja z določenim meškom, ki je povezan s asimetričnimi stroški, če ne dosežejo ali presežejo cilj.

KLJUČNE UGOTOVITVE:

- Temne trge zmanjšujejo cene in raziskovanje cene v primeru pozitivnih rizikov izvajanja.
- Cene in raziskovanje cene postanejo neskončno manjši, če se riziko izvajanja zmanjša.
- Če presežen strošek je dovolj manjši kot strošek nedospeka in vrednost privatne informacije, trgovci presežejo predvzemno v temne trge.

POMEN ZA TRGOVANJE: Raziskava je uporabna za razumevanje vpliva temnih trgov na cene in raziskovanje cene. Trgovci, ki uporabljajo temne trge, morajo upoštevati asimetrične stroške, ki so povezani z obveznostjo izvajanja. To značilnost lahko vključuje v svoje strategije za izvajanje, da bi se izognili pretežnim stroškom.

Price Discovery and Dark Pool Execution under Deadline Constraints: An Option-Pricing Approach

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This version: September 15, 2026

Abstract

In this paper, we clarify the effect of dark pools on market quality by analyzing a theoretical model in which the informed trader's venue choice is represented by an option-pricing model. Institutional traders who use dark pools are typically working an execution mandate: they must acquire a target quantity by a deadline, and they bear asymmetric costs when they fall short of or overshoot that target. We embed this constraint—a newsvendor problem in the sense of [Arrow et al. \(1951\)](#)—into a [Kyle \(1985\)](#) model with a lit exchange and a dark pool. Assuming that the opposite-side liquidity reaching the dark pool is log-normally distributed makes the expected shortfall and overshoot costs collapse to the Black–Scholes call formula, and the equilibrium can be solved in closed form. The informed trader's dark pool order is characterized by a quantile condition on the Black–Scholes delta, and the lit order emerges as the hedging demand generated by the short straddle. In equilibrium, two results are derived. First, the presence of a dark pool strictly lowers price discovery relative to the Kyle benchmark for any positive execution risk, and price discovery vanishes entirely as execution risk vanishes. Second, when the overshoot penalty is sufficiently small relative to the shortfall penalty and the value of private information, the informed trader overshoots pre-emptively into the dark pool, there is no linear equilibrium, and price discovery collapses.

Keywords: Dark pools; Price discovery; Market microstructure; Execution risk; Newsvendor problem; Option pricing.

JEL classification: G14; G12; D82.

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1. Introduction

Dark pools have grown on the back of technological progress, regulatory reform—Regulation NMS in the United States and the Markets in Financial Instruments Directive in Europe—and the broader expansion of equity trading. Their growth has been accompanied by persistent unease. Practitioners have questioned the opacity of dark pool disclosure and the conflicts of interest faced by venue operators; academics and regulators have repeatedly asked what dark trading does to the price formation process of the market as a whole. The debate has two sides. Dark pools plainly improve trading opportunities by allowing large orders to be worked without displaying intent; at the same time they may impair the fairness and efficiency of price formation, because the orders they absorb never reach the consolidated tape. This paper takes the second concern seriously and constructs a model in which the availability of a dark pool lowers the price discovery capacity of the market.

Any analysis of dark pools and market quality must take seriously the order placement strategies and the professional constraints of the traders who actually use them. Dark pools are used predominantly by institutional investors and hedge funds. These traders are, in many cases, precisely the informed traders whose order flow drives price discovery: they possess superior information production capabilities, and the way in which they route their orders determines how much of their information is impounded into prices. A model of dark trading that abstracts from their institutional constraints therefore risks mischaracterizing the very mechanism it sets out to study.

The constraint we emphasize is the execution mandate. An institutional trader charged with executing a large block wishes to avoid the price impact of trading on the lit exchange, but must weigh that saving against the execution risk of the dark pool, where a trade is never guaranteed. Crucially, the trader is not free to choose any quantity: the mandate specifies a target, and missing it is costly. We formalize this by adding two assumptions to the standard and highly tractable framework of [Kyle \(1985\)](#). First, the informed trader is assigned a target execution quantity. Second, if the realized total execution falls short of the target, the trader incurs an explicit or implicit cost proportional to the shortfall; if it exceeds the target, the trader incurs a cost at a different penalty rate. The costs of underexecution and overexecution are thus asymmetric. This is exactly the structure of the newsvendor problem formalized by [Arrow et al. \(1951\)](#). [Cont and Kukanov \(2017\)](#) bring the same formulation into the optimal execution literature, and [Ganchev et al. \(2010\)](#) identify a newsvendor structure in the problem of allocating an exogenous parent order across several dark pools.

Our methodological contribution is an assumption that makes this problem solvable. We assume that the opposite-side liquidity that the informed trader meets in the dark pool

is lognormally distributed. Under this assumption the expected newsvendor cost reduces to the Black–Scholes option pricing formula, and the equilibrium can be analyzed in closed form. More than that, the informed trader’s order placement strategy acquires the structure of an option hedging strategy, and the Greeks supply its economic interpretation. Concretely, the dark pool order can be viewed as a short straddle whose payoff is the private information gain and whose cost is the newsvendor penalty, written on the random dark pool execution with the residual mandate as the strike; the lit market order is then the hedging demand generated by that position. Treating limit orders as options is an idea with a long lineage, beginning with [Copeland and Galai \(1983\)](#); the present paper contributes to that tradition by showing that the venue choice problem itself, and not only the order type, admits an option-theoretic representation.

How well the market performs price discovery is measured by the precision with which the lit market maker estimates the fundamental value, and this depends in the end on how much of the informed trader’s order is routed to the lit exchange. Our main result is that the presence of a dark pool strictly lowers price discovery. The informativeness coefficient that governs equilibrium is bounded strictly below the value that reproduces the [Kyle \(1985\)](#) benchmark, and it approaches that benchmark only in the limit of unbounded execution risk. Furthermore, when the overexecution cost is sufficiently small relative to the underexecution cost and the value of private information, the market maker cannot form a consistent linear estimate of fundamental value at all: the informativeness coefficient turns negative, no linear equilibrium with a positive price impact exists, and the price formation mechanism breaks down.

The direction of our conclusion places the paper in direct dialogue with two benchmark studies. [Zhu \(2014\)](#) shows that the execution risk of a dark pool can *improve* lit market price discovery, because informed traders, who cluster on the same side of the market and therefore face the worst crossing odds, self-select into the exchange. We study the same mechanism but reach the opposite conclusion, because in our setting the informed trader’s use of the dark pool is disciplined not by competition from other informed traders but by an execution mandate. [Ye and Zhu \(2020\)](#) study a different mechanism, in which the dark pool executes at the exchange price, so that the informed trader’s own price impact is carried over to her dark fill and she trades less aggressively on the exchange. They point out that price discovery therefore falls below the Kyle benchmark, and falls further as the value of her information rises.

Among closely related theoretical work on the choice between dark and lit venues, [Buti et al. \(2017\)](#) is central. [Kratz and Schöneborn \(2014\)](#) approach the same trade-off from the optimal execution side, characterizing how a large trader splits a liquidation between a price-impacting exchange and a dark pool in which execution is uncertain, and [Menkveld](#)

et al. (2017) model venue choice as a pecking order along which traders sort by the urgency of their execution needs. On the empirical side, Hendershott and Jones (2005), Nimalendran and Ray (2014), Comerton-Forde and Putniņš (2015), Foley and Putniņš (2016) and Buti et al. (2022) examine the relationship between dark trading and microstructure measures of liquidity and price discovery.

The remainder of the paper is organized as follows. Section 2 sets up the model and establishes the Black–Scholes representation of the newsvendor cost. Section 3 derives the informed trader’s optimal orders and develops their option-pricing interpretation. Section 4 characterizes the market maker’s strategy, the equilibrium and price discovery. Section 5 concludes. Proofs are collected in Appendix A.

2. The model

2.1. Assets, venues and traders

Consider a static model with two dates, 0 and 1. The asset is traded at date 0 and liquidated at date 1. The fundamental value of the asset is $\tilde{v} \sim N(\bar{v}, \tau_v^2)$, and the value of $\tilde{v}$ is realized at date 1.

There are two trading venues. The first is a conventional exchange, which we refer to as the *lit* market; the second is a *dark pool*.

The informed trader has risk-neutral preferences and maximizes expected profit. She receives the realization v of the fundamental value at date 0 as private information. Here $\delta \equiv v - \bar{v}$ is her information gain. She submits an order $x_L = \alpha + \beta(v - \bar{v}) = \alpha + \beta\delta$ to the lit market and an order x_D to the dark pool, where β is her order aggressiveness. Throughout we consider the case $x_L \geq 0$ and $x_D \geq 0$.

Liquidity traders in the lit market submit a net order $\tilde{z} \sim N(0, \tau_z^2)$, independent of $\tilde{v}$. In the lit market a market maker aggregates the informed trader’s order x_L and the realization z of the liquidity traders’ order, and, conditional on the total order flow $y = x_L + z$, forms the linear estimate of the fundamental value

$$p_L = \mathbb{E}[\tilde{v} | y] = \bar{v} + \lambda_L (y - \mathbb{E}[y]) \quad (1)$$

where λ_L is the price impact, Kyle’s lambda.

In the dark pool, the informed trader’s order x_D is matched against the orders of dark pool liquidity traders, as described in Section 2.2. The dark pool uses the midquote as its reference price and executes at

$$p_D = \bar{v} \quad (2)$$

so that no price impact arises there. This is the benefit of the dark venue in the model, and

the execution risk described next is its cost.

2.2. Order matching in the dark pool

We now describe the matching technology of orders in the dark pool. When the informed trader submits an order x_D to the dark pool, we assume that the quantity returned to her as an execution is

$$\tilde{G}(x_D) = \tilde{\theta}_D h(x_D), \quad h(x_D) = 1 - e^{-x_D} \quad (3)$$

Here $\tilde{G}(x_D)$ is read as the cumulative quantity executed by an order of size x_D , and h is the fraction of the available opposite-side liquidity that the order succeeds in capturing. Whatever part of the order goes unexecuted is not re-routed to the lit market, and is cancelled. The scaling parameter $\tilde{\theta}_D$ characterizes the execution capacity of the dark pool, and we assume that it follows a lognormal distribution.

Assumption 1 (Lognormality of opposite-side liquidity). The informed trader does not observe $\tilde{\theta}_D$ but knows its objective distribution:

$$\ln \tilde{\theta}_D \sim N(m, \tau_D^2), \quad \text{that is,} \quad \tilde{\theta}_D \sim LN(m, \tau_D^2), \quad \bar{\theta}_D \equiv \mathbb{E}[\tilde{\theta}_D] = e^{m + \frac{1}{2}\tau_D^2} \quad (4)$$

with $\tau_D > 0$. Here $\bar{\theta}_D$ is average dark pool liquidity and τ_D , the standard deviation of $\ln \tilde{\theta}_D$, is what we call the *execution risk* of the dark pool.

Lognormality is not merely a convenience. Opposite-side liquidity is intrinsically non-negative, and a normal specification would assign positive probability to $\tilde{\theta}_D \leq 0$. More importantly for what follows, the lognormal family is closed under scaling, so the reproductive property of (3) carries through to the executed quantity, which can be computed in closed form.

Lemma 1 (Lognormality of dark pool execution). *Under Assumption 1 and (3), since $\tilde{G}(x_D) = \tilde{\theta}_D h(x_D)$ we have $\ln \tilde{G} = \ln \tilde{\theta}_D + \ln h(x_D)$, and because $\ln \tilde{\theta}_D \sim N(m, \tau_D^2)$ it follows that $\mathbb{E}[\ln \tilde{G}] = m + \ln h(x_D)$ and*

$$\ln \tilde{G} \sim N(m + \ln h(x_D), \tau_D^2), \quad \text{that is,} \quad \tilde{G} \sim LN(m + \ln h(x_D), \tau_D^2) \quad (5)$$

Applying this to the lognormal mean formula, the expected executed quantity is

$$\mu_G \equiv \mathbb{E}[\tilde{G}(x_D)] = e^{m + \ln h(x_D) + \frac{1}{2}\tau_D^2} = h(x_D) e^{m + \frac{1}{2}\tau_D^2} = h(x_D) \bar{\theta}_D \quad (6)$$

Differentiating (6) and using $h'(x_D) = e^{-x_D}$,

$$\frac{\partial \mu_G}{\partial x_D} = h'(x_D) \bar{\theta}_D = e^{-x_D} \bar{\theta}_D > 0, \quad \frac{\partial^2 \mu_G}{\partial x_D^2} = h''(x_D) \bar{\theta}_D = -e^{-x_D} \bar{\theta}_D < 0 \quad (7)$$

In other words, as the informed trader's dark pool order x_D increases the expected executed quantity $\mathbb{E}[\tilde{G}]$ rises, but the increase is diminishing, so that larger orders are progressively harder to execute.

2.3. The informed trader's execution cost

The informed trader's cost is a function determined by the target execution quantity and the realized execution. She must acquire a target quantity $X > 0$ by the deadline, and her cost is

$$C(x_L, \tilde{G}(x_D)) = \kappa (X - x_L - \tilde{G}(x_D))^+ + \rho (x_L + \tilde{G}(x_D) - X)^+, \quad \kappa, \rho > 0 \quad (8)$$

where $(X - x_L - \tilde{G}(x_D))^+$ denotes $\max\{0, X - x_L - \tilde{G}(x_D)\}$. If the total executed quantity falls short of the target, then $X - x_L - \tilde{G}(x_D) > 0$, the first term survives, and an under-execution penalty at rate κ is incurred. Conversely, if the total executed quantity exceeds the target, then $X - x_L - \tilde{G}(x_D) < 0$, the first term becomes zero, and an overexecution penalty at rate ρ is incurred. This is precisely the formulation of [Arrow et al. \(1951\)](#),² who examined the problem of balancing inventory costs against opportunity costs. Institutional investors may be taken to face a cost structure of this kind, and are accordingly driven by the motive to minimize execution costs (e.g., tracking error on an unfilled index trade, alpha decay before the deadline, and the cost of unwinding an overfill).

This cost function coincides with the European call and put pricing formulas of [Black and Scholes \(1973\)](#). The reason is that $\tilde{G}$ follows a lognormal distribution and that the residual mandate $D \equiv X - x_L$ can be regarded as a strike price. Since we are considering a cost rather than a payoff, the first term is analogous to holding a short position in κ units of a put option struck at D , and the second term to holding a short position in ρ units of a call option struck at the same D . The informed trader is therefore implicitly running a short straddle strategy, holding different numbers of units at a common strike price.

We make use of the identity

$$C(x_L, \tilde{G}(x_D)) = \kappa (X - x_L - \tilde{G}(x_D)) + (\kappa + \rho) (x_L + \tilde{G}(x_D) - X)^+ \quad (9)$$

²The structure in (8) is the classical *newsvendor problem*: a decision maker commits to a quantity before the relevant random variable is realized, and pays an underexecution penalty for each unit short of the target and an overexecution penalty for each unit in excess of it. We use the term throughout.

which follows from put–call parity. It replaces the kinked pair of options by a linear term plus a single call.

2.4. Moments of over/under execution cost

We now compute the moments of the over/under execution cost. Since $\tilde{G}$ is lognormal, define, for a given residual mandate $D > 0$,

$$d_- \equiv \frac{\ln \mu_G - \ln D}{\tau_D} - \frac{\tau_D}{2}, \quad d_+ \equiv \frac{\ln \mu_G - \ln D}{\tau_D} + \frac{\tau_D}{2} = d_- + \tau_D \quad (10)$$

Lemma 2 (Black–Scholes moments of the executed quantity). *Let $\tilde{G} \sim LN(m + \ln h(x_D), \tau_D^2)$ with $\mu_G = \mathbb{E}[\tilde{G}]$, and let $D > 0$. Then*

$$\Pr(\tilde{G} > D) = \Phi(d_-) \quad (11)$$

$$\mathbb{E}[(\tilde{G} - D)^+] = \mathbb{E}[\tilde{G}] \Phi(d_+) - D \Phi(d_-) \quad (12)$$

where Φ is the distribution function of the standard normal.

Expression (12) has the same structure as the Black–Scholes call option pricing formula: $\tilde{G}$ as the forward price of the underlying, D as the strike price and τ_D as the volatility of the underlying, with the discount factor e^{-rT} implicitly set to one, because we consider a static model. The economics of the execution mandate can thereby be expressed in a vocabulary that is already familiar.

This interpretation also delivers the differential structure as *Greeks*. Because $\Phi(d_{\pm})$ depends on both μ_G and D , differentiating (12) naively produces density terms; the following identity—the same one that arises when one differentiates a Black–Scholes call with respect to the price of the underlying in order to obtain the Greek delta—removes them.

Lemma 3 (Delta identity). *With $d_{\pm}$ defined in (10) and ϕ the standard normal density,*

$$\mu_G \phi(d_+) = D \phi(d_-) \quad (13)$$

Consequently, writing $\mathcal{C}(\mu_G, D) \equiv \mathbb{E}[(\tilde{G} - D)^+]$,

$$\frac{\partial \mathcal{C}}{\partial \mu_G} = \Phi(d_+), \quad \frac{\partial \mathcal{C}}{\partial D} = -\Phi(d_-) \quad (14)$$

Identity (13) is used in the algebraic manipulation and comparative statics below.

3. The informed trader's optimal order strategy

3.1. The informed trader's optimization problem

The informed trader is risk neutral and can learn the realization v of the fundamental value in advance as private information. Her objective function V is the sum of the expected payoff obtained from the lit market, the expected payoff obtained from the dark pool, and the expected cost borne from underexecution and overexecution:

$$\begin{aligned} V &= \mathbb{E}[(\tilde{v} - \tilde{p}_L)x_L + (\tilde{v} - p_D)\tilde{G}(x_D) - C(x_L, \tilde{G}(x_D)) \mid v] \\ &= (\delta - \lambda_L x_L)x_L + \delta \mathbb{E}[\tilde{G}(x_D)] - \mathbb{E}[C(x_L, \tilde{G}(x_D))] \\ &= (\delta + \kappa)x_L - \lambda_L x_L^2 - \kappa X + (\delta + \kappa) \mathbb{E}[\tilde{G}(x_D)] - (\kappa + \rho) \mathbb{E}[(\tilde{G}(x_D) - D)^+] \end{aligned} \quad (15)$$

where the second line substitutes $\tilde{p}_L = \bar{v} + \lambda_L(x_L + \tilde{z} - \mathbb{E}[y])$ from (1), and $p_D = \bar{v}$ from (2). It also uses $\mathbb{E}[\tilde{z}] = 0$ and the independence of $\tilde{z}$ from x_L . The third line uses the identity (9). Substituting Lemmas 1 and 2, the maximization problem is

$$\max_{x_D, x_L} V = (\delta + \kappa)x_L - \lambda_L x_L^2 - \kappa X + (\delta + \kappa)h(x_D)\bar{\theta}_D - (\kappa + \rho)[h(x_D)\bar{\theta}_D\Phi(d_+) - D\Phi(d_-)] \quad (16)$$

subject to

$$D = X - x_L, \quad \mu_G = \bar{\theta}_D h(x_D), \quad d_{\pm} = \frac{\ln \mu_G - \ln D}{\tau_D} \pm \frac{\tau_D}{2} \quad (17)$$

3.2. The optimal order to the dark pool

Proposition 1 (Optimal dark pool order). *Denote the informed trader's optimal strategy by (x_D^*, x_L^*) . The optimal order to the dark pool x_D^* satisfies*

$$\Phi(d_+^*) = \frac{\delta + \kappa}{\kappa + \rho} \quad (18)$$

or, equivalently,

$$\frac{\ln \mu_G^* - \ln D^*}{\tau_D} + \frac{\tau_D}{2} = \Phi^{-1}\left(\frac{\delta + \kappa}{\kappa + \rho}\right) \quad (19)$$

with $\mu_G^* = \bar{\theta}_D h(x_D^*)$ and $D^* = X - x_L^*$.

The right-hand side of (18) is a ratio whose numerator is the expected payoff from submitting one additional unit to the dark pool—the information gain δ together with the underexecution penalty κ that the additional unit avoids—and whose denominator is the expected cost incurred by that additional unit, consisting, by put–call parity, of κ together

with the overexecution penalty ρ . The left-hand side is the adjusted probability of overexecution, or more precisely the overexecution probability measured under the risk-neutral measure. The condition therefore states that the informed trader orders in the dark pool in such a way that the marginal benefit of increasing her order by one unit is balanced against the marginal cost of overexecution that it generates. It is the critical ratio of the newsvendor problem, transplanted onto the delta of an option written on her own execution.

Two limiting properties are worth recording. As $\tau_D \rightarrow 0$,

$$x_D^* \longrightarrow -\ln\left(1 - \frac{X}{e^m}\right) \quad (20)$$

where $X \leq e^m$ is required for $x_D^* \geq 0$; and if the further condition $X \rightarrow e^m$ is imposed, then $x_D^* \rightarrow \infty$. This shows that if the execution risk of the dark pool is vanishingly small, and if the dark pool has enough capacity to absorb the informed trader's execution target, then the informed trader routes without limit to the dark pool. The reason is that the dark pool carries no price impact, so that once execution risk is zero it becomes something the informed trader should exploit to the maximum.

3.3. Uniqueness of the optimal dark pool order

Proposition 2 (Uniqueness). *Let the informed trader's optimal strategy be (x_D^*, x_L^*) . For a given x_L^* with $D^* = X - x_L^* > 0$, the optimal order to the dark pool x_D^* exists uniquely.*

Proof. $\Phi(d_+^*)$ is continuous and strictly increasing in x_D^* , since

$$\frac{\partial d_+}{\partial x_D} = \frac{1}{\tau_D} \cdot \frac{e^{-x_D}}{1 - e^{-x_D}} > 0 \quad (21)$$

Hence the equation $\Phi(d_+) = (\delta + \kappa)/(\kappa + \rho)$ has at most one solution, and if we restrict attention to the economically meaningful range

$$\frac{\delta + \kappa}{\kappa + \rho} < \Phi(\bar{d}_+) \equiv \Phi\left(\lim_{x_D \rightarrow \infty} d_+\right) = \Phi\left(\frac{\ln \bar{\theta}_D - \ln D^*}{\tau_D} + \frac{\tau_D}{2}\right) \quad (22)$$

then by the intermediate value theorem exactly one interior solution exists. $\square$

3.4. The optimal order to the lit market

Proposition 3 (Optimal lit market order). *Let the informed trader's optimal strategy be (x_D^*, x_L^*) . The optimal order to the lit market x_L^* satisfies*

$$x_L^* = \frac{(\delta + \kappa) - (\kappa + \rho)\Phi(d_-^*)}{2\lambda_L} = \frac{(\kappa + \rho)[\Phi(d_+^*) - \Phi(d_-^*)]}{2\lambda_L} \quad (23)$$

where

$$d_{\pm}^* = \frac{\ln \mu_G^* - \ln(X - x_L^*)}{\tau_D} \pm \frac{\tau_D}{2}, \quad \mu_G^* = \bar{\theta}_D h(x_D^*) \quad (24)$$

Expressions (18) and (23) show that the lit market order is motivated by hedging demand against the execution risk of the dark pool. It is the spread $\Phi(d_+^*) - \Phi(d_-^*)$ between the delta and the exercise probability, weighted by the total penalty $\kappa + \rho$ and discounted by the price impact $2\lambda_L$.

The spread is itself an option-theoretic object. For small τ_D it equals $\phi(d_+^*)\tau_D$, the probability-weighted convexity premium generated by execution volatility; it is strictly positive if and only if $\tau_D > 0$, and it vanishes together with execution risk. This result yields a perspective in which the optimal execution strategy in the presence of a dark pool is an option trade on execution risk placed in the dark venue, with the lit market serving as the arena to which that position is brought back to be hedged. The classical view of limit orders as options, initiated by Copeland and Galai (1983), is thereby extended not only to the order type but also to the venue choice.

Proposition 4 (Execution risk creates the lit market). *Since $\Phi(d_+^*) - \Phi(d_-^*) > 0$, the lit market order (23) is strictly positive. As $\tau_D \rightarrow 0$ we have $d_-^* \rightarrow d_+^*$, so that $\Phi(d_-^*) \rightarrow (\delta + \kappa)/(\kappa + \rho)$ and hence $x_L^* \rightarrow 0$; for a given price impact λ_L , the lit market order is of the same order of magnitude as the execution risk τ_D .*

In other words, as the uncertainty of execution in the dark pool becomes vanishingly small, the informed trader drives her lit market order arbitrarily close to zero. This is because the dark pool has no price impact and executes at the midquote, so that in circumstances where execution is certain the advantage of the dark pool becomes overwhelming. Turned around, the proposition says that orders reach the lit market, and price discovery functions at all, only because execution risk exists in the dark pool. Note that this holds under risk neutrality; neither risk aversion nor ambiguity aversion is needed. What generates the lit market order is the convexity of the newsvendor penalty, not the curvature of the trader's utility.

Because (18) pins down d_+^* independently of x_L , the two first-order conditions can in fact be solved explicitly.

Corollary 1 (Explicit solution of the optimal orders). *In any interior optimum, $d_+^* = \Phi^{-1}((\delta + \kappa)/(\kappa + \rho))$ and $d_-^* = d_+^* - \tau_D$ depend on the primitives only through the critical ratio. Hence, for a given price impact λ_L ,*

$$x_L^* = \frac{\kappa + \rho}{2\lambda_L} [\Phi(d_+^*) - \Phi(d_+^* - \tau_D)] \quad (25)$$

$$x_D^* = -\ln[1 - (X - x_L^*) \exp(\tau_D d_+^* - m - \tau_D^2)] \quad (26)$$

4. The market maker and equilibrium

4.1. The market maker's optimal strategy

In the lit market the market maker observes the realization of the total order flow y and forms a linear estimate of the fundamental value of the asset. Differentiating the informed trader's optimal lit market order x_L^* with respect to the information gain, the market maker who takes (23) as given may set

$$\beta \equiv \frac{\partial x_L^*}{\partial \delta} = \frac{B}{2\lambda_L}, \quad \text{where} \quad B = \mathbb{E} \left[\frac{\partial}{\partial \delta} \{ (\kappa + \rho) [\Phi(d_+^*) - \Phi(d_-^*)] \} \right] \quad (27)$$

is the informativeness coefficient. Because $x_L^*(\delta)$ in (25) is not exactly affine in δ , we work with the linear projection, which is the natural counterpart of the linear pricing rule (1).³

Definition 1 (Linear equilibrium). A linear equilibrium is a price impact λ_L and a strategy $(x_L^*(\delta), x_D^*(\delta))$ such that (i) given λ_L , the strategy solves (16) for every δ , and (ii) the pricing rule (1) is the least-squares linear predictor of $\tilde{v}$ given y , evaluated at the linear projection $x_L^*(\delta) \approx \alpha + \beta \delta$ of the trader's strategy.

Since δ is normally distributed, Stein's lemma justifies the representation in (27), in the sense that $\beta = \text{Cov}(x_L^*, \delta) / \tau_v^2 = \mathbb{E}[\partial x_L^* / \partial \delta]$: the projection coefficient is the expected sensitivity of the lit market order to the information gain, with the expectation taken under the prior distribution.

Proposition 5 (Closed form for the informativeness coefficient). *In any interior optimum,*⁴

$$B = \mathbb{E} \left[1 - \exp(\tau_D d_+^* - \frac{1}{2} \tau_D^2) \right], \quad d_+^* = \Phi^{-1} \left(\frac{\delta + \kappa}{\kappa + \rho} \right) \quad (28)$$

³The projection is therefore an approximation, and the equilibrium conditions below are stated to leading order. Writing $e \equiv x_L^*(\delta) - \alpha - \beta \delta$ for the projection residual, the order flow has variance $\beta^2 \tau_v^2 + \text{Var}(e) + \tau_z^2$, so that the exact least-squares coefficient replaces τ_z^2 by $\tau_z^2 + \text{Var}(e)$ in the projection formula of Section 4.2. This leaves $\mathcal{J} = B/2$ in (31) unchanged and affects only the level of λ_L^* and the existence condition, which becomes $\tau_v^2 B(2 - B) > V_g$, where V_g is the residual variance of $g(\delta) \equiv (\delta + \kappa) - (\kappa + \rho) \Phi(d_+^* - \tau_D)$ about its linear projection. A second-order expansion of g gives $V_g = O(\tau_v^4)$, so under the small prior dispersion maintained in footnote 4 the correction is of second order and displaces the regime boundary of Theorem 2 by $O(\tau_v^2)$.

⁴The characterization (28) presumes an interior optimum, which by Proposition 2 requires $0 < (\delta + \kappa) / (\kappa + \rho) < \Phi(\bar{d}_+)$. Since δ has unbounded support, the expectation in (28) is taken over the interior region; equivalently, we assume that the prior dispersion τ_v is small relative to the penalty parameters κ and ρ , so that corner solutions carry negligible probability mass. For $\delta \leq -\kappa$ the marginal benefit of a dark pool order is non-positive and $x_D^* = 0$; for $(\delta + \kappa) / (\kappa + \rho) \geq \Phi(\bar{d}_+)$ the dark pool order is unbounded. Neither corner affects the qualitative results below.

4.2. Equilibrium

Theorem 1 (Linear equilibrium). *There exists a linear equilibrium (x_D^*, x_L^*, p_L^*) in which the price impact and the order aggressiveness are*

$$\lambda_L^* = \frac{\tau_v}{2\tau_z} \sqrt{B(2-B)}, \quad \beta^* = \frac{\tau_z}{\tau_v} \sqrt{\frac{B}{2-B}} \quad (29)$$

A real solution exists if and only if $0 < B < 2$.

By the projection theorem, $\lambda_L = \beta \tau_v^2 / (\beta^2 \tau_v^2 + \tau_z^2) = (B/2\lambda_L) \tau_v^2 / [(B/2\lambda_L)^2 \tau_v^2 + \tau_z^2]$ holds. So the fixed point for λ_L degenerates to a quadratic and is solved by a square root. At $B = 1$ the solution reproduces the [Kyle \(1985\)](#) benchmark exactly, $\lambda_L^* = \tau_v / (2\tau_z)$.

4.3. Price discovery and the existence of equilibrium

Price discovery is measured by the fraction of the prior variance of the fundamental value that the lit order flow resolves,

$$\mathcal{J} \equiv 1 - \frac{\text{Var}(\tilde{v} | y)}{\tau_v^2} = \frac{\beta^2 \tau_v^2}{\beta^2 \tau_v^2 + \tau_z^2} \quad (30)$$

Substituting the equilibrium order aggressiveness β^* of Theorem 1 gives

$$\mathcal{J} = \frac{B}{2} \quad (31)$$

so that price discovery is governed by the informativeness coefficient alone, and the [Kyle \(1985\)](#) benchmark $\mathcal{J} = 1/2$ corresponds to $B = 1$. The value $B = 1$, however, is never attained.

Proposition 6 (Strict reduction in price discovery). *For every finite $\tau_D > 0$,*

$$B = 1 - \mathbb{E}[\exp(\tau_D d_+^* - \frac{1}{2} \tau_D^2)] < 1, \quad \text{hence} \quad \mathcal{J} = \frac{B}{2} < \frac{1}{2} \quad (32)$$

The Kyle benchmark $B = 1$ holds only asymptotically, either as $\tau_D^2 \rightarrow \infty$ or as $\rho \rightarrow \infty$ with the remaining parameters fixed. At the opposite extreme, $B \rightarrow 0$ and $\mathcal{J} \rightarrow 0$ as $\tau_D \rightarrow 0$.

The result follows because the exponential inside the expectation in (28) is strictly positive, so that the expectation cannot vanish for finite execution risk. Its content is unambiguous: whenever a dark pool with positive execution risk is available, the lit market maker estimates fundamental value strictly less precisely than in the [Kyle \(1985\)](#) economy without a dark venue, and the price discovery measure lies strictly below one half. The

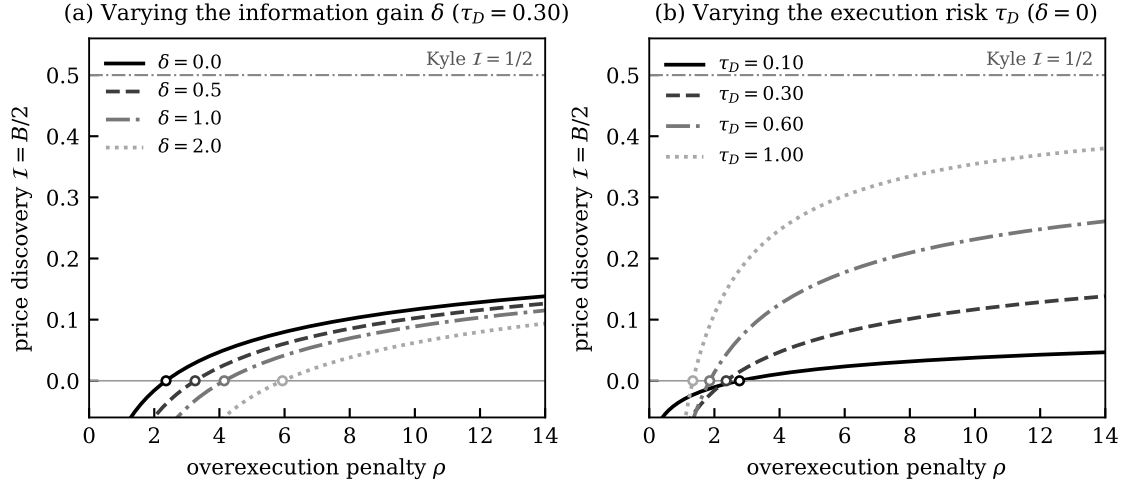


Figure 1. Price discovery $\mathcal{I} = B/2$ as a function of the overexecution penalty ρ , at $\kappa = 3$. Panel (a) varies the information gain δ holding $\tau_D = 0.30$; panel (b) varies the execution risk τ_D holding $\delta = 0$. Open circles on the horizontal axis mark the regime boundary $\rho^*(\tau_D) = (\delta + \kappa)/\Phi(\tau_D/2) - \kappa$ of Theorem 2. In both panels $\mathcal{I}$ is increasing in ρ and decreasing in δ throughout, and stays strictly below the Kyle benchmark $1/2$ everywhere. Within the regular regime—to the right of the open circles, where $d_+^* < \tau_D/2$ —it is in addition increasing in τ_D .

reduction is largest precisely when the dark pool is most attractive, that is, when execution risk is smallest. In the limit of a perfectly reliable dark pool the lit market receives no informed order flow at all and price discovery disappears entirely, consistent with Proposition 4.

Theorem 1 requires $0 < B < 2$; by Proposition 6 the upper bound never binds, so the existence of a linear equilibrium hinges entirely on the sign of B . Figure 1 plots $\mathcal{I} = B/2$ against the overexecution penalty, separately for variation in the information gain and in execution risk; price discovery rises with ρ and with τ_D , falls with δ , and approaches the Kyle benchmark only in the limit.

4.4. Regimes and the breakdown of linear price formation

The sign of B is governed by the asymmetry of the execution penalty.

Theorem 2 (Regime change). *The informativeness coefficient is positive if and only if*

$$d_+^* < \frac{\tau_D}{2} \iff \frac{\delta + \kappa}{\kappa + \rho} < \Phi\left(\frac{\tau_D}{2}\right) \iff \rho > \rho^*(\tau_D) \quad (33)$$

where the boundary in the (ρ, τ_D) plane is

$$\rho^*(\tau_D) \equiv \frac{\delta + \kappa}{\Phi(\tau_D/2)} - \kappa \quad (34)$$

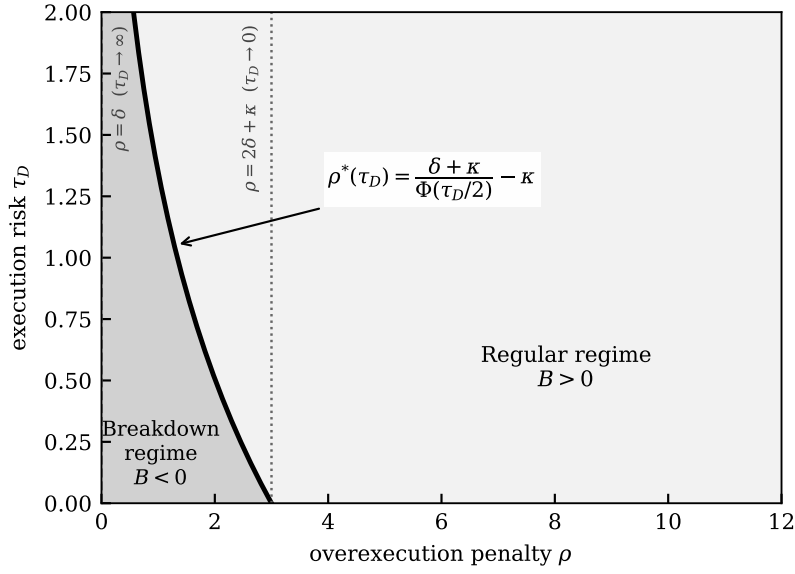


Figure 2. The regular and breakdown regimes in the (ρ, τ_D) plane, drawn at $\delta = 0$ and $\kappa = 3$. The solid curve is the boundary (34); the shaded region to its left is the breakdown regime, in which $B < 0$ and no linear equilibrium with a positive price impact exists. The boundary is decreasing in τ_D and lies between the asymptotes $\rho = 2\delta + \kappa$ and $\rho = \delta$ of Corollary 2.

We call the region $\rho > \rho^*(\tau_D)$, in which $B > 0$ and a linear equilibrium with a positive price impact exists, the regular regime; and the region $\rho < \rho^*(\tau_D)$, in which $B < 0$, $\sqrt{B(2-B)}$ is not real, and no linear equilibrium with a positive price impact exists, the breakdown regime.

Corollary 2 (Shape of the boundary). $\rho^*(\tau_D)$ is strictly decreasing in τ_D , with

$$\rho^*(0) = 2\delta + \kappa, \quad \lim_{\tau_D \rightarrow \infty} \rho^*(\tau_D) = \delta \quad (35)$$

The breakdown region therefore contracts as execution risk rises, and the leading-order threshold $\rho = 2\delta + \kappa$ is an upper bound on the boundary, attained only in the limit $\tau_D \rightarrow 0$.

Corollary 2 says that execution risk and the overexecution penalty are substitutes in sustaining a linear equilibrium: a dark pool whose execution is more uncertain tolerates a weaker overexecution penalty before price formation fails. The gap between the exact boundary and its leading-order limit is not negligible at empirically plausible magnitudes: at $\kappa = 3$, $\delta = 0$ and $\tau_D = 0.30$ the exact boundary is $\rho^* = 2.36$ against the limiting value 3.00. Figure 2 maps the two regimes in the (ρ, τ_D) plane. Because the boundary falls monotonically from $2\delta + \kappa$ towards δ , the breakdown regime is confined to a wedge of low overexecution penalties that narrows as the dark pool becomes less reliable.

The intuition for the breakdown regime is direct. When the overexecution penalty ρ is small, the informed trader fears underexecution and has an incentive to place a pre-

emptively excessive order in the dark pool; and the stronger her private information, the more she overshoots. The order flow that remains in the lit market is then negatively correlated with the fundamental value, $\beta^* = \partial x_L^* / \partial \delta < 0$, and a competitive market maker would have to quote a negative price impact to be consistent with it. The linear pricing rule becomes self-contradictory and the equilibrium fails to exist. The model thus suggests that, depending on the market environment, the behaviour of informed traders can bring the price formation function of the lit market to the point of collapse. Note that what disappears is the linear equilibrium, not the market itself; whether a non-linear equilibrium survives in the breakdown regime is beyond the scope of the linear pricing rule adopted here.

Condition (33) is stated pointwise in δ , whereas B in (28) is an expectation under the prior, so strictly the sign of B depends on where the bulk of the prior mass lies relative to the boundary. Panel (a) of Figure 1 shows that the pointwise object varies only gently with δ over the relevant range, so that under the small prior dispersion maintained in footnote 4 the expected and pointwise conditions deliver the same regime.

4.5. Comparative statics

Every equilibrium object is a function of the informativeness coefficient alone, and B in turn depends on the primitives only through the critical ratio $r = (\delta + \kappa) / (\kappa + \rho)$ and the execution risk τ_D , so the comparative statics collapse onto the response of $d_+^* = \Phi^{-1}(r)$. Proposition 7 in Appendix A derives them in closed form, in terms of the pointwise informativeness $b(\delta) \equiv 1 - \exp(\tau_D d_+^* - \frac{1}{2} \tau_D^2)$, which satisfies $B = \mathbb{E}[b(\delta)]$ and, under the small prior dispersion maintained in footnote 4, coincides with B to leading order. Table 1 collects the signs. Five readings deserve comment.

First, the four lit-order derivatives in (40) are themselves option Greeks. The responses to the two penalty rates are the call and the put value per unit of residual mandate; the response to the information gain is the pointwise informativeness itself; and the response to execution risk has the form of a vega. The hedging reading of Section 3.4 therefore extends to the comparative statics and not only to the level of the order.

Second, ρ and κ push the lit order in the same direction but price discovery in opposite directions. Both penalties make the newsvendor position more expensive at the margin and so raise x_L^* ; but a larger ρ lowers the critical ratio while a larger κ raises it, because a dark fill also buys release from the shortfall penalty. Since what the market maker learns from is the *sensitivity* of the lit order to δ rather than its level, only the second channel reaches price discovery, and the two signs separate. The sign of $\partial b / \partial \kappa$ is that of $-(\rho - \delta)$ and is unambiguous only once the regular regime is imposed; Corollary 2 gives $\rho^*(\tau_D) > \delta$ for every τ_D , which is what secures it.

Third, better private information hurts price discovery. This reverses the Kyle (1985)

Table 1. Comparative statics. Entries give the sign of the partial derivative of the column object with respect to the row primitive, evaluated at an interior optimum in the regular regime $\rho > \rho^*(\tau_D)$; $\pm$ marks a derivative whose sign is not determined there. Following Section 3, the two order columns are evaluated at a given price impact λ_L , and the remaining columns are equilibrium objects. The δ row is pointwise in δ , as in Theorem 2.

		x_D^*	x_L^*	B	$\mathcal{J} = B/2$	λ_L^*	β^*
τ_D	execution risk	−	+	+	+	+	+
ρ	overexecution penalty	−	+	+	+	+	+
κ	underexecution penalty	$\pm$	+	−	−	−	−
X	target quantity	+	0	0	0	0	0
$\bar{\theta}_D$	dark pool depth	−	0	0	0	0	0
δ	information gain	$\pm$	+	−	−	−	−

intuition, in which a larger information gain makes the informed trader trade more aggressively on the exchange. Here the larger δ does raise x_L^* , by (40), but it raises the critical ratio at the same time, so each additional unit of information is routed disproportionately into the dark; pushed far enough this is exactly what produces the breakdown regime of Theorem 2. The same asymmetry moves the regime boundary: $\partial\rho^*/\partial\delta$ and $\partial\rho^*/\partial\kappa$ are both positive by (34), so stronger information and a heavier shortfall penalty widen the breakdown wedge, while $\partial\rho^*/\partial\tau_D < 0$ narrows it.

Fourth, λ_L^* and β^* move together rather than against each other. In the one-period Kyle model a deeper noise pool lowers λ and raises β along a fixed frontier; here τ_v/τ_z is held fixed and what shifts is the quantity of private information that reaches the exchange at all, so a larger B raises the adverse selection the market maker faces and the aggressiveness of the trader at the same time.

Fifth, the size of the mandate and the depth of the dark pool move the dark order alone—a larger target raises x_D^* , a deeper pool lowers it, since less needs to be submitted to obtain the same expected fill—and leave every equilibrium object untouched. The trader adjusts x_D^* to restore the ratio μ_G^*/D^* required by (18), and x_L^* is fixed by the marginal trade-off between price impact and the newsvendor penalty, in which neither X nor $\bar{\theta}_D$ enters. Both matter for whether the optimum is interior at all, and hence for which regime the market is in, but not for price discovery within a regime. This is why the empirical design sketched in Section 5 is built on variation in the penalty rates rather than on variation in target quantities.

5. Concluding remarks

This paper analyzes the effect of a dark pool on the price discovery capacity of the lit market using a Kyle model extended to incorporate the execution constraints specific to in-

stitutional investors. An informed trader who is assigned a target execution quantity bears asymmetric costs for underexecution and overexecution. This structure is a newsvendor problem, and assuming a lognormal distribution for opposite-side liquidity reduces the expected cost to the Black–Scholes call price formula, which makes it possible to solve the equilibrium in closed form. The formulation also permits the informed trader’s order placement strategy and venue choice to be reinterpreted as an option trade: a short straddle written in the dark pool, and its hedging demand executed on the lit market.

Two results stand out. First, for any positive execution risk the informativeness coefficient lies strictly below unity, so that price discovery is strictly worse than in the [Kyle \(1985\)](#) benchmark; and as execution risk vanishes, informed order flow deserts the lit market altogether and price discovery disappears. Order flow reaches the exchange only because execution in the dark is uncertain. Second, when the overexecution penalty is small relative to the underexecution penalty and the value of private information, stronger information induces the trader to reduce rather than increase her lit market order; the informativeness coefficient changes sign, no linear equilibrium exists, and the market maker cannot estimate the fundamental value.

These results should be interpreted with care. Since a single informed trader diverts order flow to a venue that the market maker cannot observe, the *direction* of the deterioration is mechanically transparent. The contribution lies rather in the finding that the *magnitude* of the deterioration is proportional to execution risk, and that a qualitatively distinct breakdown regime appears on the other side of the critical ratio, along a boundary that contracts as execution risk rises. By contrast, if one places many informed traders with a common signal in the model, as in [Zhu \(2014\)](#), their clustering on the same side of the dark pool raises execution risk and generates self-selection into the exchange, a channel that works in the direction of improving price discovery. Which of the two forces dominates can be assessed by extending the present framework to many traders, and we leave this for future work.

Several other extensions suggest themselves: moving from the static single period to a multi-period setting; allowing unexecuted dark pool orders to be re-routed to the lit market; and introducing risk aversion or ambiguity aversion into the informed trader’s preferences. On the last point, whether dark pool usage rises or falls depends on whether the object of ambiguity is the opposite-side liquidity of the dark pool or the price impact of the lit market, and which effect dominates remains an open question.

Finally, the model yields testable predictions. At index additions and deletions, and at futures expiries, the cost of falling short of the mandate is likely to be large, since tracking error and the need to rebuild positions make underexecution expensive. Such events therefore supply variation in the underexecution penalty κ relative to the overexecution

penalty ρ , and hence, through the critical ratio $(\delta + \kappa)/(\kappa + \rho)$, in the position of the regime boundary of Theorem 2. Exploiting these situations together with microstructure measures of dark trading volume and price discovery, one can identify the relationship between execution constraints and market quality that the model describes.

A. Proofs

Proof of Lemma 2

Write $\tilde{G} = e^W$ with $W \sim N(\mu, \tau_D^2)$ and $\mu = m + \ln h(x_D)$, so that $\ln \mu_G = \mu + \tau_D^2/2$. Then

$$\Pr(\tilde{G} > D) = \Pr(W > \ln D) = \Phi\left(\frac{\mu - \ln D}{\tau_D}\right) = \Phi(d_-)$$

since $d_- = (\ln \mu_G - \ln D)/\tau_D - \tau_D/2 = (\mu - \ln D)/\tau_D$. Next,

$$\mathbb{E}[\tilde{G} \mathbf{1}\{\tilde{G} > D\}] = \int_{\ln D}^{\infty} e^w \frac{1}{\tau_D \sqrt{2\pi}} e^{-\frac{(w-\mu)^2}{2\tau_D^2}} dw$$

Completing the square in the exponent factors out $e^{\mu+\tau_D^2/2} = \mu_G$ and leaves a normalized Gaussian integral over $(\ln D, \infty)$ whose mean is shifted by τ_D^2 . That integral equals $\Phi((\mu + \tau_D^2 - \ln D)/\tau_D)$, which is $\Phi(d_- + \tau_D) = \Phi(d_+)$, so that $\mathbb{E}[\tilde{G} \mathbf{1}\{\tilde{G} > D\}] = \mu_G \Phi(d_+)$. Subtracting $D \Pr(\tilde{G} > D)$ gives (12). $\blacksquare$

Proof of Lemma 3

From (10), $d_+ + d_- = 2(\ln \mu_G - \ln D)/\tau_D$ and $d_+ - d_- = \tau_D$, so

$$\frac{\phi(d_+)}{\phi(d_-)} = \exp\left(-\frac{d_+^2 - d_-^2}{2}\right) = \exp\left(-\frac{(d_+ + d_-)(d_+ - d_-)}{2}\right) = \exp\left(-\ln \frac{\mu_G}{D}\right) = \frac{D}{\mu_G}$$

which is (13). For the derivatives, note $\partial d_{\pm}/\partial \mu_G = 1/(\mu_G \tau_D)$ and $\partial d_{\pm}/\partial D = -1/(D \tau_D)$.

Hence

$$\begin{aligned} \frac{\partial \mathcal{C}}{\partial \mu_G} &= \Phi(d_+) + \frac{1}{\mu_G \tau_D} [\mu_G \phi(d_+) - D \phi(d_-)] = \Phi(d_+) \\ \frac{\partial \mathcal{C}}{\partial D} &= -\Phi(d_-) - \frac{1}{D \tau_D} [\mu_G \phi(d_+) - D \phi(d_-)] = -\Phi(d_-) \end{aligned}$$

where the bracketed terms vanish by (13). $\blacksquare$

Proof of Proposition 1

Differentiating (16) with respect to x_D and using $\partial\mu_G/\partial x_D = h'(x_D)\bar{\theta}_D$ together with $\partial\mathcal{C}/\partial\mu_G = \Phi(d_+)$ from Lemma 3,

$$\frac{\partial V}{\partial x_D} = (\delta + \kappa) h'(x_D) \bar{\theta}_D - (\kappa + \rho) h'(x_D) \bar{\theta}_D \Phi(d_+) = 0$$

and since $h'(x_D)\bar{\theta}_D > 0$ the result (18) follows. The second-order condition holds at any such point, because

$$\frac{\partial^2 V}{\partial x_D^2} = -(\kappa + \rho) h'(x_D) \bar{\theta}_D \phi(d_+) \frac{\partial d_+}{\partial x_D} < 0$$

by (21), evaluated where the bracketed term vanishes.

For the limit (20), note that as $\tau_D \rightarrow 0$ we have $\bar{\theta}_D \rightarrow e^m$ and, by Proposition 4, $x_L^* \rightarrow 0$ so that $D^* \rightarrow X$. Keeping d_+^* finite as $\tau_D \rightarrow 0$ requires $\ln \mu_G^* - \ln D^* \rightarrow 0$, that is $h(x_D^*)e^m = X$, which gives $x_D^* \rightarrow -\ln(1 - X/e^m)$. ■

Proof of Proposition 3 and Corollary 1

Differentiating (16) with respect to x_L , noting $\partial\mu_G/\partial x_L = 0$ and $\partial D/\partial x_L = -1$, and using $\partial\mathcal{C}/\partial D = -\Phi(d_-)$ from Lemma 3, gives

$$\frac{\partial V}{\partial x_L} = (\delta + \kappa) - 2\lambda_L x_L - (\kappa + \rho) \Phi(d_-) = 0$$

which is the first equality in (23); the objective is strictly concave in x_L because $-2\lambda_L < 0$ and $-(\kappa + \rho) \partial\Phi(d_-)/\partial x_L = -(\kappa + \rho)\phi(d_-)/(D\tau_D) < 0$. Substituting $(\delta + \kappa) = (\kappa + \rho)\Phi(d_+^*)$ from (18) gives the second equality. Since the delta identity (13) is a property of the probability distribution alone, and not of the optimization problem, it may be used as often as required in these manipulations.

For Corollary 1, (18) fixes d_+^* independently of x_L , and $d_-^* = d_+^* - \tau_D$ by (10); substituting into (23) yields (25). Inverting the definition of d_+^* ,

$$\mu_G^* = D^* \exp\left(\tau_D d_+^* - \frac{1}{2} \tau_D^2\right), \quad \text{so} \quad h(x_D^*) = \frac{D^*}{\bar{\theta}_D} \exp\left(\tau_D d_+^* - \frac{1}{2} \tau_D^2\right) = D^* e^{\tau_D d_+^* - m - \tau_D^2/2}$$

using $\bar{\theta}_D = e^{m+\tau_D^2/2}$. Solving $h(x_D^*) = 1 - e^{-x_D^*}$ for x_D^* gives (26). ■

Proof of Proposition 4

Positivity is immediate from (23) because Φ is strictly increasing and $d_+^* - d_-^* = \tau_D > 0$. For the order of magnitude, a first-order Taylor expansion of Φ around d_+^* gives $\Phi(d_+^*) - \Phi(d_+^* - \tau_D) = \phi(d_+^*)\tau_D + O(\tau_D^2)$, and d_+^* does not depend on τ_D , so that, for a given price impact λ_L , $x_L^* = \Theta(\tau_D)$ and $x_L^* \rightarrow 0$. ■

Proof of Proposition 5

From (23) and (18),

$$2\lambda_L x_L^* = (\delta + \kappa) - (\kappa + \rho)\Phi(d_+^* - \tau_D), \quad d_+^* = \Phi^{-1}\left(\frac{\delta + \kappa}{\kappa + \rho}\right)$$

Differentiating with respect to δ and using $\partial d_+^* / \partial \delta = 1 / [(\kappa + \rho)\phi(d_+^*)]$,

$$\frac{\partial (2\lambda_L x_L^*)}{\partial \delta} = 1 - \frac{\phi(d_+^* - \tau_D)}{\phi(d_+^*)}$$

The ratio of densities is

$$\frac{\phi(d_+^* - \tau_D)}{\phi(d_+^*)} = \exp\left(-\frac{(d_+^* - \tau_D)^2 - (d_+^*)^2}{2}\right) = \exp(\tau_D d_+^* - \frac{1}{2}\tau_D^2)$$

Taking the expectation over the prior distribution and applying (27) gives (28). ■

Proof of Theorem 1 and equation (31)

Substituting $\beta = B/(2\lambda_L)$ into $\lambda_L = \beta\tau_v^2/(\beta^2\tau_v^2 + \tau_z^2)$ and multiplying through by $\lambda_L(\beta^2\tau_v^2 + \tau_z^2)$ gives

$$\frac{B^2}{4}\tau_v^2 + \lambda_L^2\tau_z^2 = \frac{B}{2}\tau_v^2 \quad \Longleftrightarrow \quad \lambda_L^2\tau_z^2 = \frac{\tau_v^2}{4}B(2-B)$$

so $\lambda_L^* = \frac{\tau_v}{2\tau_z}\sqrt{B(2-B)}$, which is real if and only if $B(2-B) > 0$, that is $0 < B < 2$. Then $\beta^* = B/(2\lambda_L^*) = \frac{\tau_z}{\tau_v}\sqrt{B/(2-B)}$ and $\beta^{*2}\tau_v^2 = \tau_z^2 B/(2-B)$, so that

$$\mathcal{J} = \frac{\beta^{*2}\tau_v^2}{\beta^{*2}\tau_v^2 + \tau_z^2} = \frac{B/(2-B)}{B/(2-B) + 1} = \frac{B}{2} \quad \blacksquare$$

Proof of Proposition 6

By (28), $B = 1 - \mathbb{E}\left[e^{\tau_D d_+^* - \tau_D^2/2}\right]$. The random variable inside the expectation is strictly positive, so its expectation is strictly positive and $B < 1$ for every finite τ_D ; hence $\mathcal{J} = B/2 < 1/2$. As $\tau_D^2 \rightarrow \infty$ with d_+^* fixed, the exponent $\tau_D d_+^* - \tau_D^2/2 \rightarrow -\infty$ and $B \rightarrow 1$; the

same limit obtains as $\rho \rightarrow \infty$ with δ , κ and τ_D fixed, since then $(\delta + \kappa)/(\kappa + \rho) \rightarrow 0$ and $d_+^* \rightarrow -\infty$. As $\tau_D \rightarrow 0$ the exponent tends to zero, so $B \rightarrow 0$ and $\mathcal{J} \rightarrow 0$. ■

Proof of Theorem 2 and Corollary 2

The integrand in (28) is positive if and only if $\tau_D d_+^* - \tau_D^2/2 < 0$, that is $d_+^* < \tau_D/2$, which by monotonicity of Φ is $(\delta + \kappa)/(\kappa + \rho) < \Phi(\tau_D/2)$. Solving this inequality for ρ gives $\rho > (\delta + \kappa)/\Phi(\tau_D/2) - \kappa = \rho^*(\tau_D)$, which is (34). If $B \leq 0$ then $B(2 - B) \leq 0$ and, by Theorem 1, λ_L^* has no positive real solution.

For Corollary 2, $\Phi(\tau_D/2)$ is strictly increasing in τ_D and $\delta + \kappa > 0$, so $\rho^*(\tau_D)$ is strictly decreasing. Evaluating at the endpoints, $\Phi(0) = 1/2$ gives $\rho^*(0) = 2(\delta + \kappa) - \kappa = 2\delta + \kappa$, and $\Phi(\tau_D/2) \rightarrow 1$ as $\tau_D \rightarrow \infty$ gives $\rho^*(\tau_D) \rightarrow \delta + \kappa - \kappa = \delta$. ■

Comparative statics: Proposition 7

Throughout, write the pointwise informativeness as

$$b(\delta) \equiv 1 - \exp(\tau_D d_+^* - \frac{1}{2} \tau_D^2), \quad B = \mathbb{E}[b(\delta)] \quad (36)$$

which under the small prior dispersion maintained in footnote 4 coincides with B to leading order.

Proposition 7 (Comparative statics). *Write $e^u \equiv \exp(\tau_D d_+^* - \frac{1}{2} \tau_D^2) > 0$.*

(a) Informativeness and price discovery.

$$\frac{\partial b}{\partial \rho} = \frac{\tau_D (\delta + \kappa)}{(\kappa + \rho)^2 \phi(d_+^*)} e^u > 0, \quad \frac{\partial b}{\partial \kappa} = -\frac{\tau_D (\rho - \delta)}{(\kappa + \rho)^2 \phi(d_+^*)} e^u, \quad (37)$$

$$\frac{\partial b}{\partial \delta} = -\frac{\tau_D}{(\kappa + \rho) \phi(d_+^*)} e^u < 0, \quad \frac{\partial b}{\partial \tau_D} = -(d_+^* - \tau_D) e^u \quad (38)$$

In the regular regime $\rho > \rho^(\tau_D) > \delta$ and $d_+^* < \tau_D/2 < \tau_D$, so $\partial b/\partial \kappa < 0$ and $\partial b/\partial \tau_D > 0$ there. On the admissible range $0 < B < 1$ of Proposition 6,*

$$\frac{\partial \lambda_L^*}{\partial B} = \frac{\tau_v}{2\tau_z} \cdot \frac{1 - B}{\sqrt{B(2 - B)}} > 0, \quad \frac{\partial \beta^*}{\partial B} = \frac{\tau_z}{\tau_v} \cdot \frac{1}{\sqrt{B(2 - B)^{3/2}}} > 0, \quad \frac{\partial \mathcal{J}}{\partial B} = \frac{1}{2} \quad (39)$$

so λ_L^ , β^* and $\mathcal{J}$ inherit the sign of the response of B .*

(b) Order quantities. *Let $\mathcal{C} \equiv \mathbb{E}[(\tilde{G} - D^*)^+]$ and $\mathcal{P} \equiv \mathbb{E}[(D^* - \tilde{G})^+] = \mathcal{C} - (\mu_G^* - D^*)$*

be the call and put values of Section 2.4. For a given price impact λ_L ,

$$2\lambda_L \frac{\partial x_L^*}{\partial \rho} = \frac{\mathcal{C}}{D^*}, \quad 2\lambda_L \frac{\partial x_L^*}{\partial \kappa} = \frac{\mathcal{P}}{D^*}, \quad 2\lambda_L \frac{\partial x_L^*}{\partial \delta} = b, \quad 2\lambda_L \frac{\partial x_L^*}{\partial \tau_D} = (\kappa + \rho) \phi(d_-^*) \quad (40)$$

all four of which are strictly positive in the regular regime, and x_L^* does not depend on X or $\bar{\theta}_D$. The dark pool order satisfies

$$h(x_D^*) = \frac{D^*}{\bar{\theta}_D} \exp(\tau_D d_+^* - \frac{1}{2} \tau_D^2), \quad D^* = X - x_L^* \quad (41)$$

with h strictly increasing, so that $\partial x_D^*/\partial X > 0$, $\partial x_D^*/\partial \bar{\theta}_D < 0$, $\partial x_D^*/\partial \rho < 0$ and $\partial x_D^*/\partial \tau_D < 0$, while $\partial x_D^*/\partial \kappa$ and $\partial x_D^*/\partial \delta$ are of ambiguous sign.

Proof. Part (a). Differentiating $r = (\delta + \kappa)/(\kappa + \rho)$ gives $\partial r/\partial \rho = -(\delta + \kappa)/(\kappa + \rho)^2$, $\partial r/\partial \kappa = (\rho - \delta)/(\kappa + \rho)^2$ and $\partial r/\partial \delta = 1/(\kappa + \rho)$, while $\partial d_+^*/\partial r = 1/\phi(d_+^*)$ and d_+^* does not depend on τ_D . Since $b = 1 - e^u$ with $u = \tau_D d_+^* - \frac{1}{2} \tau_D^2$, we have $\partial b/\partial \xi = -e^u \tau_D \partial d_+^*/\partial \xi$ for $\xi \in \{\rho, \kappa, \delta\}$ and $\partial b/\partial \tau_D = -e^u (d_+^* - \tau_D)$, which is (37)–(38). By Corollary 2, $\rho^*(\tau_D) = (\delta + \kappa)/\Phi(\tau_D/2) - \kappa > \delta$ because $\Phi(\tau_D/2) < 1$, so $\rho > \rho^*(\tau_D)$ implies $\rho > \delta$; Theorem 2 gives $d_+^* < \tau_D/2$ in the regular regime. Differentiating (29) yields (39), both expressions being positive for $0 < B < 1$.

Part (b). Write $2\lambda_L x_L^* = (\delta + \kappa) - (\kappa + \rho)\Phi(d_-^*)$ from (23), with $d_-^* = d_+^* - \tau_D$. The derivative with respect to δ is $1 - \phi(d_-^*)/\phi(d_+^*) = b$, as in the proof of Proposition 5. For ρ , using $\partial d_+^*/\partial \rho$ from part (a) and $\phi(d_-^*)/\phi(d_+^*) = e^u = \mu_G^*/D^*$,

$$\frac{\partial (2\lambda_L x_L^*)}{\partial \rho} = -\Phi(d_-^*) + \frac{\phi(d_-^*)}{\phi(d_+^*)} \cdot \frac{\delta + \kappa}{\kappa + \rho} = \frac{\mu_G^*}{D^*} \Phi(d_+^*) - \Phi(d_-^*) = \frac{\mathcal{C}}{D^*}$$

because $(\delta + \kappa)/(\kappa + \rho) = \Phi(d_+^*)$ by (18) and $\mathcal{C} = \mu_G^* \Phi(d_+^*) - D^* \Phi(d_-^*)$ by (12). For κ , the same substitutions with $(\rho - \delta)/(\kappa + \rho) = 1 - \Phi(d_+^*)$ give

$$\frac{\partial (2\lambda_L x_L^*)}{\partial \kappa} = 1 - \Phi(d_-^*) - \frac{\mu_G^*}{D^*} (1 - \Phi(d_+^*)) = \frac{\mathcal{C} - (\mu_G^* - D^*)}{D^*} = \frac{\mathcal{P}}{D^*}$$

the last equality being put–call parity. For τ_D , d_+^* is fixed and $\partial d_-^*/\partial \tau_D = -1$, so the derivative is $(\kappa + \rho)\phi(d_-^*)$. All four are strictly positive, $\mathcal{C}$ and $\mathcal{P}$ because $\tau_D > 0$ makes $\tilde{G}$ non-degenerate, and b by Theorem 2. Neither X nor $\bar{\theta}_D$ appears in (23), (18) or (28), which gives the stated invariance.

Equation (41) is (26) rearranged, and since $h' > 0$ the sign of $\partial x_D^*/\partial \xi$ is that of $\partial \ln h(x_D^*)/\partial \xi$, where

$$\ln h(x_D^*) = \ln(X - x_L^*) + \tau_D d_+^* - \frac{1}{2} \tau_D^2 - \ln \bar{\theta}_D$$

Differentiating, $\partial \ln h / \partial X = 1/D^* > 0$ and $\partial \ln h / \partial \bar{\theta}_D = -1/\bar{\theta}_D < 0$. For ρ both terms are negative, since x_L^* rises and d_+^* falls; for τ_D , holding $\bar{\theta}_D$ fixed, the sum is $-(1/D^*)\partial x_L^* / \partial \tau_D + d_+^* - \tau_D$, negative because $d_+^* < \tau_D/2$. For κ and δ the two terms carry opposite signs, so no determinate sign obtains. ■

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