

7461079-two-sided-volatility-jumps-in-short-maturity-vix-options

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POVZETEK:

Članek raziskuje model, ki omogoča dvostranske volatilnostsko skokove v kratkoterminnih VIX opcijah. Raziskovalci razvijajo mehanizem, ki je opredeljen v stanju in generira dvostranske skokove volatilnosti, ki so povezane z volatilnostskimi skokoma v akcijskem cijenju. Negativni skoki akcijske cijene povečujejo volatilnost, dokler pozitivni skoki akcijske cijene jo smanjujejo. Volatilnostsko skokovsko vrednost je multiplikativna in povezana z predskokovno vrednostjo. Podobno analizirajo vrednosti opcij s različnimi stopnji volatilnosti in pokazujejo, da dvostranski model omogoča pozitivne vrednosti za oba tipa opcij, če je volatilnostsko skokovsko vrednost pozitivna.

METODA:

Raziskovalci razvijajo risk-neutralen stanje-zavisni zaznamani Hawkesov tip, ki je opredeljen v stanju in generira dvostranske skokove volatilnosti. Volatilnostsko skokovsko vrednost je povezana z predskokovno vrednostjo akcijske cijene. Negativni skoki akcijske cijene povečujejo volatilnost, pozitivni pa jo smanjujejo. Volatilnostsko skokovsko vrednost je multiplikativna in povezana z predskokovno vrednostjo. Stopnja volatilnostskih skokov je omejena in zagotavlja pozitivno stopnjo skokov akcijske cijene.

KLJUČNE UGOTOVITVE:

- Negativni skoki akcijske cijene povečujejo volatilnost, pozitivni pa jo smanjujejo.
- Volatilnostsko skokovsko vrednost je multiplikativna in povezana z predskokovno vrednostjo.
- Dvostranski model omogoča pozitivne vrednosti za oba tipa opcij, če je volatilnostsko skokovsko vrednost pozitivna.
- Volatilnostsko skokovsko vrednost je omejena in zagotavlja pozitivno stopnjo skokov akcijske cijene.

POMEN ZA TRGOVANJE:

Raziskave pokazeta, da dvostranski model volatilnosti skokov v kratkoterminnih VIX opcijah omogoča pravilnejši ocenjevanje vrednosti opcij. To je posebno pomembno za trgovce, ki uporabljajo VIX opcije za ocenjevanje rizika volatilnosti. Pomembno je, da model omogoča pravilno ocenjevanje vrednosti oba tipa opcij, če je volatilnostsko skokovsko vrednost pozitivna.

Two-Sided Volatility Jumps in Short-Maturity VIX Options*

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Abstract

VIX options pose a distinctive modeling problem: accommodating downward VIX jumps while preserving a positive VIX level and capturing the inverse relation between stock returns and VIX changes. We develop a risk-neutral state-dependent marked Hawkes-type mechanism in which the pre-jump intensity and signed stock-jump mark jointly determine the intensity update. Negative stock jumps generate excitation and upward VIX jumps, whereas positive stock jumps generate de-excitation and downward VIX jumps. Multiplicative VIX jumps preserve a positive VIX level, and a local admissible stopped construction keeps the stock-jump intensity nonnegative. Thus, a single marked stock-jump process generates two-sided VIX jumps without an independent VIX jump process. Under a local linear-in-intensity squared-VIX approximation, we derive explicit short-maturity option-price expansions. With positive current intensity and fixed out-of-the-money strikes, calls and one-jump-reachable puts have strictly positive leading terms proportional to maturity; the two benchmarks allowing only upward VIX jumps instead yield put values of smaller order. The nearly flat within-expiration-cycle maturity profile of put–call ratios at symmetric absolute log-moneyness is consistent with equal-order short-end put and call pricing.

Keywords: Two-sided volatility jumps; State-dependent marked Hawkes-type processes; Short-maturity VIX options; Small-maturity option asymptotics; Volatility tail risk.

JEL classification codes: C22, C58, C60, G12, G13.

Subject classification: Financial Engineering; Derivatives; Point processes; Asymptotic analysis

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1. Introduction

VIX options pose a distinctive modeling problem. The VIX is a nontradable, nonnegative index computed from a strip of S&P 500 option prices and embeds a fixed 30-day variance horizon, even when the option itself expires within days. VIX changes are negatively correlated with S&P 500 returns, motivating a jump mechanism that links stock-price declines to upward VIX jumps and stock-price increases to downward VIX jumps. The challenge is to accommodate downside VIX jumps while preserving a positive VIX level and a nonnegative stock-jump intensity. Models with exclusively upward volatility jumps capture VIX spikes but exclude this downward VIX-jump channel. We address this problem through a state-dependent marked Hawkes-type construction that generates two-sided VIX jumps.

The short-maturity VIX option data make this problem economically consequential. Across 514 weekly expiration cycles, the farthest reported out-of-the-money (OTM) put buckets finish in the money in approximately 10–13% of the 1-, 4-, 5-, and 7-day quote-date samples, compared with approximately 7–8% for the farthest reported OTM call buckets. Both wings carry positive quoted prices and substantial trading activity. These cash-settlement crossing frequencies highlight the economic relevance of short-horizon downside VIX risk and motivate a model that incorporates downward VIX jumps.

Our contribution is a signed stock-jump-dependent update of the jump-arrival intensity under the risk-neutral measure. A local approximation links squared VIX linearly to that intensity. We specify an inverse relation between stock and VIX log jumps and derive the intensity update that preserves this local relation. A negative stock-jump mark generates excitation and raises VIX, whereas a positive stock-jump mark generates de-excitation and lowers VIX. The VIX-level jump is multiplicative: a downward VIX jump scales the pre-jump level by a factor strictly between zero and one, preserving a strictly positive VIX level.

An upper bound on positive stock-jump marks, together with a local admissibility condition and stopping rule, keeps the stock-jump intensity nonnegative. Thus, a single marked stock-jump process produces both VIX jump directions without introducing an independent VIX jump process. The intensity update depends jointly on the pre-jump intensity and the contemporaneous stock-jump mark.

The pricing implication is an order-of-decay separation. Let Δ denote the VIX-option maturity. Under the local approximation and stopped construction, with positive current jump intensity, we derive explicit leading-order prices for fixed-strike OTM VIX options. Every such call and every such put whose strike can be crossed by one admissible downward VIX jump have strictly positive leading terms proportional to Δ . Under their respective local conditions, the two benchmarks allowing only upward VIX jumps instead yield $o(\Delta)$ put values as $\Delta \downarrow 0$, while one-jump-reachable calls retain positive order- Δ values. For put and call strikes with equal absolute log-moneyness relative to current VIX, the put–call ratio therefore converges to a finite, strictly positive limit on the two-sided model’s one-jump-reachable put set. In contrast, it converges to zero under a one-sided benchmark whenever the call has a positive order- Δ coefficient.

We empirically examine this order comparison using the ratio of OTM VIX put and call prices at equal absolute log-moneyness. Across specifications, the estimated within-expiration-cycle maturity profile is nearly flat, indicating little variation in the relative price of puts and calls over the observed short maturities. This pattern is consistent with VIX put and call option prices decaying at the same leading order as expiration approaches, as implied by the two-sided VIX jump mechanism.

2. Evidence on short-horizon VIX strike-crossing risk

Table 1 summarizes 514 weekly VIX option expiration cycles from October 28, 2015, through August 27, 2025. For the 1-, 4-, 5-, and 7-day quote-date buckets, it reports payoff-positive strike-crossing frequencies, Black-implied volatilities, offer prices, trading volume, and open interest. For a selected strike K , a payoff-positive crossing occurs when $VIX_{t+\Delta} < K$ for a put and when $VIX_{t+\Delta} > K$ for a call. Here $VIX_{t+\Delta}$ is the VIX Special Opening Quotation used for cash settlement.

The days-to-expiration (DTE) labels use the quote-date convention for expiring weekly VIX options. For the standard weekly cycle, the one-day bucket uses the Monday close as the quote time, Tuesday is ordinarily the last trading day, and settlement ordinarily occurs Wednesday morning; the four-, five-, and seven-day buckets use the preceding Friday, Thursday, and Tuesday closes, respectively. Thus, the DTE label describes the quote date, whereas Δ measures the interval from the quote close to cash settlement, not

to the last traded quote.

The displayed contracts are actively quoted and traded: average daily volume is in the thousands of contracts and average open interest is in the tens of thousands. The put bucket closest to at-the-money crosses 233, 280, 282, and 298 times in the one-, four-, five-, and seven-day samples, respectively, whereas the farthest reported OTM put bucket crosses 53, 63, 65, and 50 times. Because the reported frequencies use 514 expiration cycles, the latter counts correspond to payoff-positive frequencies of approximately 10–13%, compared with approximately 7–8% for the farthest reported OTM calls. The farthest OTM put buckets also have reported offer prices of \$0.14–\$0.22, average daily volume of 2,511–3,368 contracts, and average open interest of 19,920–22,731 contracts.

These realized strike-crossing frequencies highlight the economic relevance of short-horizon downside VIX risk and motivate a specification permitting both upward and downward VIX movements.

3. Small-maturity asymptotics for VIX option pricing and benchmarks

Section 3.1 constructs the two-sided VIX–stock jump link. On an admissible interval, the log-contract identity motivates a local linear-in-intensity squared-VIX proxy, constructed through a mean-field closure and local linearization. The pricing analysis uses fixed positive proxy coefficients, a jump-time admissibility condition, and a stopped state process to maintain nonnegative intensity. Section 3.2 compares the resulting option asymptotics with two one-sided benchmarks.

3.1. A two-sided state-dependent marked Hawkes-type VIX–stock jump link

We use the following notation: (i) S_t is the S&P 500 index level; (ii) r is the constant interest rate, and the model abstracts from dividends; (iii) $V > 0$ is the constant diffusive variance component; (iv) W_t^s is a standard Brownian motion; (v) N_t counts stock jumps and has predictable intensity λ_{t-} ; and (vi) J_t^s is the stock log-jump mark at jump time τ_t . We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ satisfying the usual conditions, and $\mathbb{E}_t^{\mathbb{Q}}[\bullet]$ denotes conditional expectation given $\mathcal{F}_t$ under the risk-neutral measure $\mathbb{Q}$. On every horizon for which the stock model is invoked, we assume that the discounted stock price is a true $\mathbb{Q}$ -martingale and that the expected integrated jump intensity is finite.

Together with the truncated-normal mark moments, these conditions make the log-contract expectations finite. Let $\text{VIX}_{t+\Delta}$ denote the 30-day VIX level at VIX option expiration. Here Δ is the VIX-option maturity, whereas $\Delta^* := \frac{30}{365}$ is the fixed variance horizon embedded in the VIX definition.

Stock index dynamics and stock jump marks. Under the risk-neutral measure $\mathbb{Q}$, the stock index follows

$$\frac{dS_t}{S_{t-}} = r dt + \sqrt{V} dW_t^S + (e^{J_t^S} - 1) dN_t - \lambda_{t-} \mathbb{E}^{\mathbb{Q}}[e^{J^S} - 1] dt, \quad (1)$$

where the expectation $\mathbb{E}^{\mathbb{Q}}[e^{J^S} - 1]$ is taken with respect to the mark law conditional on a jump. In (1), $(e^{J_t^S} - 1)dN_t$ denotes the stock-price jump generated by the realized mark at time t . Conditional on $\mathcal{F}_{\tau_i-}$ and on a stock jump at τ_i , the mark law is independent of the pre-jump state and is upper-truncated normal:

$$J_i^S \sim \text{Normal}(m_J, \sigma_J^2) \text{ upper truncated to } (-\infty, J_{\max}], \quad \sigma_J > 0, \quad \text{i.i.d. under } \mathbb{Q}. \quad (2)$$

Let $F_J(dj)$ denote the upper-truncated normal law in (2). Equivalently, with $z := \frac{J_{\max} - m_J}{\sigma_J}$ and with $\Phi(\bullet)$ and $\phi(\bullet)$ denoting the standard normal cumulative distribution function and probability density function, respectively, $F_J(dj) = \frac{1}{\sigma_J \Phi(z)} \phi\left(\frac{j - m_J}{\sigma_J}\right) \mathbf{1}_{\{j \leq J_{\max}\}} dj$. The following truncated-normal moments are used:

$$\mathbb{E}^{\mathbb{Q}}[e^{J^S}] = \exp\left(m_J + \frac{\sigma_J^2}{2}\right) \frac{\Phi(z - \sigma_J)}{\Phi(z)} \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}[J^S] = m_J - \sigma_J \frac{\phi(z)}{\Phi(z)}. \quad (3)$$

At a jump time τ_i , the stock index jumps by

$$S_{\tau_i} = S_{\tau_i-} e^{J_i^S}, \quad \text{equivalently} \quad \log\left(\frac{S_{\tau_i}}{S_{\tau_i-}}\right) = J_i^S. \quad (4)$$

State-dependent marked Hawkes-type intensity with signed intensity-update mark. Stock-index jumps arrive according to a state-dependent marked Hawkes-type specification. Let $\mathcal{N}(dt, dj) = \sum_i \delta_{\{\tau_i, J_i^S\}}(dt, dj)$ denote the marked jump measure ($\delta_{\{x\}}$ is the Dirac measure at x), and let $N_t = \mathcal{N}((0, t] \times (-\infty, J_{\max}])$. Up to the exit time defined below, $\mathcal{N}$ has predictable compensator $\lambda_{t-} F_J(dj) dt$.

With this notation, the jump and compensator terms in (1) can be written as

$$\int_{(-\infty, J_{\max}]} (e^j - 1) \{ \mathcal{N}(dt, dj) - \lambda_{t-} F_J(dj) dt \}. \quad (5)$$

On the same local interval, the intensity state satisfies

$$d\lambda_t = \beta(\mu - \lambda_{t-}) dt + \int_{(-\infty, J_{\max}]} A(\lambda_{t-}, j) \mathcal{N}(dt, dj), \quad \text{or, equivalently,} \quad (6)$$

$$\lambda_t = \mu + (\lambda_0 - \mu) e^{-\beta t} + \int_{(0, t] \times (-\infty, J_{\max}]} e^{-\beta(t-\ell)} A(\lambda_{\ell-}, j) \mathcal{N}(d\ell, dj), \quad (7)$$

where $\mu \geq 0$ is the baseline jump intensity and $\beta > 0$ governs exponential decay toward μ between jumps.

In equations (6)–(7), the signed state-dependent intensity-update mark $A(\lambda_{\tau_i-}, J_i^s)$ depends on the pre-jump state and the contemporaneous stock-index jump; at τ_i , $\lambda_{\tau_i} - \lambda_{\tau_i-} = A(\lambda_{\tau_i-}, J_i^s)$.

The form of the intensity-update mark $A(\lambda_{\tau_i-}, J_i^s)$ is derived below by requiring consistency between the induced jump in the local VIX proxy and the squared-VIX approximation. Negative stock-jump marks generate excitation, whereas positive stock-jump marks generate de-excitation through the update in equation (13).¹ Because the update can be negative, Lemma 1 supplies a jump-time admissibility buffer and Theorem 1 works with the resulting stopped local construction.

Squared-VIX approximation and induced VIX-level jump. On any admissible continuation of (1) over $[t, t + \Delta^*]$ for which λ remains a nonnegative predictable intensity and the log-contract expectation is finite, and suppressing the conventional VIX scaling factor of 100, the log-contract representation gives $\text{VIX}_t^2 = \mathbb{E}_t^{\mathbb{Q}}[\{\frac{-2}{\Delta^*}\} \log(\frac{S_{t+\Delta^*}}{S_t e^{r\Delta^*}})]$. Hence,

$$\text{VIX}_t^2 = V + \frac{2}{\Delta^*} \times \mathbb{E}^{\mathbb{Q}}[e^{J^s} - 1 - J^s] \times \mathbb{E}_t^{\mathbb{Q}}[\int_t^{t+\Delta^*} \lambda_{\ell-} d\ell]. \quad (8)$$

Motivated by (8), Appendix A combines a mean-field closure with a first-order linearization to construct the following local linear-in- λ_t squared-VIX approximation, conditional on an admissible continuation over the fixed VIX horizon:

$$\text{VIX}_t^2 \approx b_0 + b_\lambda \lambda_t \quad (\text{using a mean-field approximation (Appendix A.2)}). \quad (9)$$

For the small-maturity VIX option-pricing analysis, we take $X_u := b_0 + b_\lambda \lambda_u$ as the local squared-VIX proxy and set $Y_u := \sqrt{X_u}$. We identify Y_t with the current model VIX level.

¹Holding the diffusive variance component V fixed isolates the stock-jump intensity channel for VIX movements. De-excitation can be interpreted as uncertainty resolution: a positive stock jump lowers the risk-neutral stock-jump intensity on impact.

The coefficients (b_0, b_λ) are held fixed throughout the option expansion and satisfy $b_0 > 0$ and $b_\lambda > 0$. Theorem 1 requires only these coefficient restrictions; when the coefficients are obtained from Appendix A.2, the theorem is conditional on the selected mean-field solution. On the local region $\lambda_u \geq 0$, positivity of X_u follows from $b_0 > 0$ and $b_\lambda > 0$.

To encode the inverse relation between stock-index and VIX movements, we specify the log jump of the local VIX proxy to be linear in the negative stock log jump. At a jump time τ , impose

$$J^{\text{vix}} := \log\left(\frac{Y_\tau}{Y_{\tau-}}\right) = -v J^s \in [-v J_{\max}, +\infty), \quad v > 0. \quad (10)$$

To preserve the local proxy relation $Y_u = \sqrt{b_0 + b_\lambda \lambda_u}$ across a jump, choose the signed state-dependent intensity-update mark $A(\lambda_{\tau-}, J_i^s)$ so that

$$\frac{Y_\tau}{Y_{\tau-}} = \exp(J^{\text{vix}}) = e^{-v J^s}. \quad (11)$$

Substituting $Y_u = \sqrt{b_0 + b_\lambda \lambda_u}$ into (11) gives

$$\frac{\sqrt{b_0 + b_\lambda (\lambda_{\tau-} + A(\lambda_{\tau-}, J^s))}}{\sqrt{b_0 + b_\lambda \lambda_{\tau-}}} = e^{-v J^s}. \quad (12)$$

Solving (12) yields the signed state-dependent intensity-update mark:

$$A(\lambda_{\tau-}, J^s) = \frac{b_0 + b_\lambda \lambda_{\tau-}}{b_\lambda} (e^{-2v J^s} - 1), \quad \text{which can be positive or negative.} \quad (13)$$

For fixed J^s , $\frac{\partial A(\lambda_{\tau-}, J^s)}{\partial \lambda_{\tau-}} = e^{-2v J^s} - 1$. Hence a larger pre-jump state increases the positive excitation when $J^s < 0$ and increases the magnitude of the negative de-excitation when $J^s > 0$.

This model also implies the following convex-asymmetry identity for equal-magnitude stock-index jumps:

$$\frac{A(\lambda_{\tau-}, -\mathbb{X})}{-A(\lambda_{\tau-}, \mathbb{X})} = \frac{e^{2v \mathbb{X}} - 1}{1 - e^{-2v \mathbb{X}}} = e^{2v \mathbb{X}} > 1, \quad \mathbb{X} > 0. \quad (14)$$

Thus, the excitation induced by a negative stock jump $J^s = -\mathbb{X}$ dominates the magnitude of the de-excitation induced by a positive stock jump $J^s = \mathbb{X}$.

We implement the signed state-dependent update in (13) through a local stopped construction. Lemma 1 supplies the jump-time admissibility buffer, and Theorem 1 derives option-price expansions for the associ-

ated stopped VIX proxy. Because the update depends on both the pre-jump state and the contemporaneous mark, the specification is Hawkes-type rather than a standard linear Hawkes process; see Bondi, Pulido, and Scotti (2024) and Laub, Lee, and Pollett (2025).

Lemma 1 (Jump-time admissibility under bounded positive stock jumps) *Under $b_0 > 0$ and $b_\lambda > 0$, assume $v > 0$ and $J_{\max} > 0$. Suppose $J^s \leq J_{\max}$ almost surely. Define the exit threshold*

$$\lambda_{\text{exit}} := \frac{b_0}{b_\lambda} (e^{2vJ_{\max}} - 1). \quad (15)$$

On $\{\tau < \infty\}$, if $\lambda_{\tau-} \geq \lambda_{\text{exit}}$, then the post-jump intensity $\lambda_\tau = \lambda_{\tau-} + A(\lambda_{\tau-}, J_\tau^s) \geq 0$.

Proof. See Appendix B.

The threshold λ_{exit} is a jump-time buffer, not an invariant lower boundary. It guarantees nonnegative intensity immediately after an admissible jump whose pre-jump state is at least λ_{exit} . The between-jump drift preserves nonnegativity when $\mu \geq 0$ but may move the state below the buffer. For the option expansion, we also impose a finite upper localization cap to bound the pre-exit jump intensity. Any jump that causes the first exit through either boundary is included before the state is frozen, preserving the full one-jump payoff support.

Small-maturity VIX option asymptotics. For strike K , the time- t values of European VIX options are

$$\text{Put}_t(K, \Delta) = e^{-r\Delta} \mathbb{E}_t^{\mathbb{Q}}[(K - \text{VIX}_{t+\Delta})^+] \quad \text{and} \quad \text{Call}_t(K, \Delta) = e^{-r\Delta} \mathbb{E}_t^{\mathbb{Q}}[(\text{VIX}_{t+\Delta} - K)^+]. \quad (16)$$

We consider the small-maturity limit $\Delta \downarrow 0$ and focus on OTM put strikes $K < \text{VIX}_t$ and OTM call strikes $K > \text{VIX}_t$. Throughout this limit, the strike K is fixed as $\Delta \downarrow 0$. We write $f(\Delta) \sim g(\Delta)$ when $\lim_{\Delta \downarrow 0} f(\Delta)/g(\Delta) = 1$, and define the one-jump put reachability lower bound $K_{\min} := \text{VIX}_t e^{-vJ_{\max}}$.

Theorem 1 is stated for a stopped local construction. Fix a finite upper localization level $\lambda_{\text{cap}} > \lambda_{\text{exit}}$. Starting from the conditioning time t , construct the state recursively as a piecewise-deterministic marked point process: while λ lies in the localized region $(\lambda_{\text{exit}}, \lambda_{\text{cap}})$, it follows $d\lambda_u = \beta(\mu - \lambda_u) du$, jumps arrive with predictable intensity λ_{u-} , their marks have law F_J , and the update in (13) is applied. Define the first boundary hit or jump exit and the associated stopped processes by

$$\tau_{\text{exit}} := \inf \{u > t : \lambda_u \notin (\lambda_{\text{exit}}, \lambda_{\text{cap}})\}, \quad N_u^{\text{stop}} := N_{u \wedge \tau_{\text{exit}}}, \quad \lambda_u^{\text{stop}} := \lambda_{u \wedge \tau_{\text{exit}}}, \quad Y_u^{\text{stop}} := Y_{u \wedge \tau_{\text{exit}}}. \quad (17)$$

Here $u \wedge \tau_{\text{exit}} = \min\{u, \tau_{\text{exit}}\}$. The exit state is recorded and then held fixed. Under the right-continuous convention, a jump that causes the first exit is included in N^{stop} and Y^{stop} ; in particular, an excitation jump that overshoots λ_{cap} is not censored.

The bound λ_{cap} on the active jump intensity makes the recursive stopped construction nonexplosive. For every Borel set $B \subseteq (-\infty, J_{\max}]$, the stopped marked count $\mathcal{N}((t, u \wedge \tau_{\text{exit}}] \times B)$ has compensator $\int_t^{u \wedge \tau_{\text{exit}}} \lambda_{\ell-} F_J(B) d\ell$. Thus the frozen state λ^{stop} is not interpreted as the counting intensity after exit. Replacing the terminal VIX in (16) by the stopped proxy gives $\text{Put}_t^{\text{stop}}(K, \Delta) = e^{-r\Delta} \mathbb{E}_t^{\mathbb{Q}}[(K - Y_{t+\Delta}^{\text{stop}})^+]$ and $\text{Call}_t^{\text{stop}}(K, \Delta) = e^{-r\Delta} \mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta}^{\text{stop}} - K)^+]$.

On the local admissible region, $Y_u = \sqrt{b_0 + b_\lambda \lambda_u}$ is piecewise deterministic: between jumps, λ_u follows $d\lambda_u = \beta(\mu - \lambda_u) du$, and jumps in Y_u occur only at stock-jump times. At the conditioning time, $Y_t^{\text{stop}} = Y_t = \text{VIX}_t$.

Theorem 1 (Small-maturity asymptotics for the stopped local VIX-level approximation) *Suppose:*

- C-1. *The coefficients (b_0, b_λ) are fixed throughout the option expansion and satisfy $b_0 > 0$ and $b_\lambda > 0$.*
- C-2. *The jump-link parameters satisfy $v > 0$, $J_{\max} > 0$, and $\mu \geq 0$. Fix $\lambda_{\text{cap}} > \lambda_{\text{exit}}$, and suppose the conditioning state lies strictly inside the localized region: $\lambda_{\text{exit}} + \chi_{\text{exit}} \leq \lambda_t \leq \lambda_{\text{cap}} - \chi_{\text{exit}}$ for some margin $\chi_{\text{exit}} > 0$.*
- C-3. *The stock-jump mark J_i^s is independent of the pre-jump state and, conditional on $\mathcal{F}_{\tau_i-}$, has the upper-truncated normal law in (2). In particular, $\mathbb{E}^{\mathbb{Q}}[e^{-q v J^s}] < \infty$ for every finite $q > 0$.*

The bounded pre-exit intensity makes the stopped count locally nonexplosive and implies $\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta}^{\text{stop}} - N_t^{\text{stop}} \geq 2) = O(\Delta^2)$. Together with C-3, it also supplies the payoff moments used below. Then the localized time- t prices of fixed-strike OTM VIX options satisfy the asymptotic expansions as $\Delta \downarrow 0$:

$$\text{Put}_t^{\text{stop}}(K, \Delta) = \begin{cases} o(\Delta), & 0 < K \leq K_{\min}, \\ \frac{e^{-r\Delta} \lambda_t \Delta}{\Phi(z)} \left(K(\Phi(z) - \Phi(d_2)) - \text{VIX}_t e^{m_{\text{vix}}} + \frac{1}{2} \sigma_{\text{vix}}^2 (\Phi(z + \sigma_{\text{vix}}) - \Phi(d_1)) \right) + o(\Delta), & K_{\min} < K < \text{VIX}_t. \end{cases} \quad (18)$$

$$\text{Call}_t^{\text{stop}}(K, \Delta) = \frac{e^{-r\Delta} \lambda_t \Delta}{\Phi(z)} \left(\text{VIX}_t e^{m_{\text{vix}}} + \frac{1}{2} \sigma_{\text{vix}}^2 \Phi(d_1) - K \Phi(d_2) \right) + o(\Delta), \quad K > \text{VIX}_t, \quad (19)$$

where $\Phi(\bullet)$ denotes the standard normal cumulative distribution function and

$$d_2 := \frac{\log(\frac{\text{VIX}_t}{K}) + m_{\text{vix}}}{\sigma_{\text{vix}}} \quad \text{and} \quad d_1 := d_2 + \sigma_{\text{vix}} = \frac{\log(\frac{\text{VIX}_t}{K}) + m_{\text{vix}} + \sigma_{\text{vix}}^2}{\sigma_{\text{vix}}}, \quad (20)$$

with $z = \frac{J_{\max} - m_J}{\sigma_J}$, $m_{\text{vix}} := -\nu m_J$, and $\sigma_{\text{vix}} := \nu \sigma_J$. Condition C-2 implies $\lambda_t > 0$. The call coefficient is therefore strictly positive for every fixed $K > \text{VIX}_t$, while the put coefficient is strictly positive exactly on the one-jump-reachable range $K_{\min} < K < \text{VIX}_t$. On those strike sets, the corresponding option price is asymptotically equivalent to its displayed order- Δ term.

Proof. See Appendix C.

Theorem 1 isolates the order comparison: every fixed OTM call and every one-jump-reachable OTM put have order- Δ value, with both leading coefficients proportional to the current jump-arrival intensity λ_t ; put strikes at or below $K_{\min}$ have no one-jump term.

3.2. Benchmark limitations in small-maturity VIX option asymptotics

We compare the two-sided construction with two one-sided benchmarks: a fixed-increment linear Hawkes model and a VIX model with strictly upward variance jumps.

3.2.1. Fixed-increment Hawkes benchmark

The fixed-increment linear Hawkes benchmark is $d\lambda_t = \beta(\mu - \lambda_{t-})dt + A_L dN_t$, with $\beta > A_L > 0$ and $\mu > 0$. Here N counts stock-jump arrivals and has predictable intensity λ_{t-} ; each arrival increases the right-continuous intensity state by A_L , irrespective of the signed stock-jump mark.

For this benchmark only, let $J_i^s \sim \text{Normal}(m_J, \sigma_J^2)$ be untruncated, with $\sigma_J > 0$. Conditional on a stock-jump arrival and the pre-jump filtration, this mark law is independent of the pre-jump state. We retain the stock dynamics in (1). The comparison is structural rather than nested: the benchmark changes both the mark law and the intensity update, replacing equation (13) by the deterministic positive increment A_L .

Then

$$\text{VIX}_t = \sqrt{w_0 + w_\lambda \lambda_t}, \quad (\text{see Appendix D}) \quad (21)$$

where $w_0 = V + \frac{2\beta\mu}{\beta - A_L}(e^{m_J + \frac{1}{2}\sigma_J^2} - 1 - m_J)(1 - \frac{1 - e^{-(\beta - A_L)\Delta^*}}{(\beta - A_L)\Delta^*})$ and $w_\lambda = 2(e^{m_J + \frac{1}{2}\sigma_J^2} - 1 - m_J)\frac{1 - e^{-(\beta - A_L)\Delta^*}}{(\beta - A_L)\Delta^*}$. Both w_0 and w_λ coefficients are strictly positive. Thus every stock-jump arrival raises VIX, even when its stock-

jump mark is positive.

Theorem 2 (Small-maturity asymptotics for the fixed-increment linear Hawkes benchmark) *In the fixed-increment benchmark specified above, with VIX mapping (21) and finite current state $0 \leq \lambda_t < \infty$, the following expansions hold for fixed positive OTM VIX-option strikes as $\Delta \downarrow 0$:*

$$\text{Put}_t(K, \Delta) = o(\Delta) \quad \text{for } K < \text{VIX}_t. \quad (22)$$

$$\text{Call}_t(K, \Delta) = e^{-r\Delta} \lambda_t \Delta (\sqrt{w_0 + w_\lambda (\lambda_t + A_L)} - K)^+ + o(\Delta), \quad K > \text{VIX}_t. \quad (23)$$

If $\lambda_t > 0$ and $K < \sqrt{w_0 + w_\lambda (\lambda_t + A_L)}$, the call price is asymptotically equivalent to the displayed order- Δ term. If $K \geq \sqrt{w_0 + w_\lambda (\lambda_t + A_L)}$, the one-jump call coefficient in (23) is zero and the fixed-strike OTM call price is $o(\Delta)$. When $\lambda_t = 0$, every fixed-strike OTM call price is also $o(\Delta)$.

Proof. See Appendix D.

The fixed-increment benchmark has no order- Δ fixed-strike OTM put term and has a finite limiting one-jump VIX reachability threshold. Define $K_A^*(t) := \sqrt{w_0 + w_\lambda (\lambda_t + A_L)}$. The leading call coefficient in equation (23) is $\lambda_t (K_A^*(t) - K)^+$, which has a kink at $K = K_A^*(t)$ when $\lambda_t > 0$. The zero put–call ratio limit in Remark 1 uses this positive-intensity, one-jump-reachable call case; when $K \geq K_A^*(t)$, the call coefficient vanishes.

3.2.2. One-sided variance-jump benchmark

Under $\mathbb{Q}$, we use the variance-jump specification and VIX mapping from Eraker and Yang (2022, equations (20)–(22) and (36)):

$$\text{Variance: } d\sigma_t^2 = \kappa_V (\theta_V - \sigma_t^2) dt + \sigma_V \sqrt{\sigma_t^2} dB_t^V + \xi_V dN_t^V, \quad (24)$$

$$\text{Jump size: } \xi_V \sim \text{Exponential}(\mu_J), \quad \xi_V > 0, \quad \mathbb{E}^\mathbb{Q}[\xi_V] = \mu_J < \infty, \quad \mathbb{E}^\mathbb{Q}[e^{u\xi_V}] < \infty, \quad \text{for } 0 \leq u < \frac{1}{\mu_J}, \quad (25)$$

$$\text{Jump intensity: } d\lambda_t^V = \kappa_\lambda (\theta_\lambda - \lambda_t^V) dt + \sigma_\lambda \sqrt{\lambda_t^V} dB_t^\lambda, \quad 2\kappa_\lambda \theta_\lambda > \sigma_\lambda^2, \quad \kappa_\lambda > 0, \theta_\lambda > 0, \sigma_\lambda > 0, \quad (26)$$

$$\text{VIX level: } \text{VIX}_t = \sqrt{a + c \times \sigma_t^2 + d \times \lambda_t^V}, \quad \text{for constants } a, c, d > 0, \quad (27)$$

where N^V is a counting process with predictable intensity $\lambda_{t-}^V \geq 0$, and B_t^V, B_t^λ are standard Brownian motions that may be correlated. The variance parameters are finite and satisfy $\kappa_V \theta_V \geq 0$ and $\sigma_V \geq 0$,

and we condition on finite nonnegative current states $(\sigma_t^2, \lambda_t^V)$. Conditional on a jump and on the pre-jump filtration, the variance-jump marks are i.i.d. with density $q(\xi) = \mu_J^{-1} e^{-\xi/\mu_J} \mathbf{1}_{\{\xi > 0\}}$; equivalently, the marked variance-jump measure has compensator $\lambda_t^V q(\xi) d\xi dt$,

The benchmark argument uses only (27), the positivity of a, c, d , and these local mark and intensity properties, not the explicit closed forms of the VIX coefficients. Let $X_t = a + c \times \sigma_t^2 + d \times \lambda_t^V$ and $\text{VIX}_t = \sqrt{X_t}$. At any variance-jump time t ,

$$X_t - X_{t-} = c\xi_V > 0 \implies \text{VIX}_t - \text{VIX}_{t-} = \sqrt{X_{t-} + c\xi_V} - \sqrt{X_{t-}} > 0. \quad (28)$$

Here, X_{t-} and VIX_{t-} denote the levels immediately before the jump, whereas X_t and VIX_t denote the levels immediately after it. Thus, in this benchmark, jumps can only raise VIX; any fixed-strike OTM put payoff must arise from the continuous component.

For the fixed-strike small-maturity asymptotic expansion, we impose the following maintained local continuous-crossing condition. Let $Y^{\text{continuous}}$ denote the VIX process generated from (24)–(27) over $[t, t + \Delta]$ from the same current state and with the same Brownian motions, after suppressing jumps of N^V . For every fixed $K \neq \text{VIX}_t$, $\mathbb{E}_t^{\mathbb{Q}}[(K - Y_{t+\Delta}^{\text{continuous}})^+] = o(\Delta)$ when $K < \text{VIX}_t$ and $\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta}^{\text{continuous}} - K)^+] = o(\Delta)$ when $K > \text{VIX}_t$. This fixed-gap condition is used only to separate the order- Δ jump contribution from the continuous diffusion contribution. A sufficient local condition is that, after suppressing jumps, the current state $(\sigma_t^2, \lambda_t^V)$ lies in a compact subset of $(0, \infty)^2$, the diffusion coefficients induced by (24)–(27) are locally bounded and locally Lipschitz on a neighborhood of that compact set, and the strike gap $|K - \text{VIX}_t|$ is strictly positive. Together with the conditional moment bounds supplied by (24)–(26), these conditions give the fixed-gap crossing estimate. Appendix E also bounds the expected correction from the immediate post-jump payoff to the terminal one-jump payoff by $O(\Delta^{3/2})$, over the full mark support.

Theorem 3 (Small-maturity asymptotics for the one-sided variance-jump benchmark) *In the model defined by (24)–(27), under the fixed-gap continuous-crossing condition and local regularity stated above, the following limits hold for fixed-strike OTM VIX options as $\Delta \downarrow 0$:*

$$\text{Put}_t(K, \Delta) = o(\Delta) \quad \text{for } K < \text{VIX}_t. \quad (29)$$

$$\text{Call}_t(K, \Delta) = e^{-r\Delta} \lambda_t^V \Delta \exp\left(\frac{\text{VIX}_t^2}{c\mu_J}\right) \frac{\sqrt{\pi c\mu_J}}{2} \text{erfc}\left(\frac{K}{\sqrt{c\mu_J}}\right) + o(\Delta), \quad K > \text{VIX}_t, \quad (30)$$

where $\text{erfc}(\bullet)$ denotes the complementary error function and $\mu_J > 0$ is the mean variance-jump size. If $\lambda_t^V > 0$, the call coefficient in (30) is strictly positive for every fixed $K > \text{VIX}_t$. Hence the call price is asymptotically equivalent to the displayed order- Δ term.

Proof. See Appendix E.

Theorem 3 characterizes the fixed-strike OTM regime, with the conditioning state and the strictly positive strike gap held fixed as $\Delta \downarrow 0$.

Together, the benchmarks have no positive order- Δ coefficient for a fixed OTM put. They retain a positive order- Δ call coefficient only on the one-jump-reachable call set: this set is bounded in Theorem 2, while the unbounded positive support of ξ_V makes every fixed $K > \text{VIX}_t$ reachable in Theorem 3 when $\lambda_t^V > 0$. This order contrast motivates the symmetric-strike diagnostic in Section 4.

4. An empirical option-surface diagnostic

The order contrast in Theorems 1, 2, and 3 motivates a reduced-form diagnostic of the relative short-maturity orders of VIX put and call prices at symmetric OTM strikes. Fix a proportional moneyness parameter $\mathbb{k} > 1$ and define

$$K_P = \text{VIX}_t \times \frac{1}{\mathbb{k}} \quad (\text{OTM put strike}) \quad \text{and} \quad K_C = \text{VIX}_t \times \mathbb{k} \quad (\text{OTM call strike}). \quad (31)$$

For the asymptotic comparison, the conditioning state at time t and both strikes are held fixed as $\Delta \downarrow 0$. By construction, $K_P < \text{VIX}_t < K_C$ and the strikes have equal absolute log-moneyness, $|\log(K_P/\text{VIX}_t)| = |\log(K_C/\text{VIX}_t)| = \log(\mathbb{k}) > 0$.² For the symmetric strike pair (K_P, K_C) , define

$$\mathcal{D}(\mathbb{k}, \Delta) := \frac{\text{Put}_t(K_P, \Delta)}{\text{Call}_t(K_C, \Delta)}. \quad (32)$$

²The theoretical order comparison uses model-specific one-jump reachability conditions. A strictly positive order- Δ put coefficient in the two-sided model requires $K_P > K_{\min} = \text{VIX}_t e^{-V_{J_{\max}}}$, equivalently $\mathbb{k} < e^{V_{J_{\max}}}$; the corresponding call condition in the fixed-increment benchmark is $K_C < K_A^*(t)$. The empirical exercise summarizes observed symmetric-strike ratios using contemporaneous VIX moneyness; structural reachability and the VIX-futures basis remain outside the regression. This spot-VIX convention matches the conditioning state in the asymptotic theory.

Remark 1 (Symmetric-strike order diagnostic) Fix $\mathbb{k} > 1$ and the symmetric OTM strikes in (31). Maintain the assumptions of the theorem corresponding to each row. For Theorem 1, suppose $\lambda_t > 0$ and $K_P > K_{\min}$, and form the ratio from the theorem's stopped local put and call prices. For Theorem 2, suppose $\lambda_t > 0$ and distinguish whether K_C is below the one-step reachability bound $K_A^*(t) = \sqrt{w_0 + w_\lambda(\lambda_t + A_L)}$. For Theorem 3, suppose $\lambda_t^V > 0$. Then, as $\Delta \downarrow 0$,

Specification	Put order	Call order	$\mathcal{D}(\mathbb{k}, \Delta)$
Theorem 1: Two-sided stopped Hawkes proxy	$\sim \alpha^{\text{put}} \Delta, \alpha^{\text{put}} > 0$	$\sim \alpha^{\text{call}} \Delta, \alpha^{\text{call}} > 0$	$\rightarrow \frac{\alpha^{\text{put}}}{\alpha^{\text{call}}} \in (0, \infty)$
Theorem 2: Fixed-increment Hawkes ($K_C < K_A^*(t)$)	$o(\Delta)$	$\sim \alpha_{T_2}^{\text{call}} \Delta, \alpha_{T_2}^{\text{call}} > 0$	$\rightarrow 0$
Theorem 2: Fixed-increment Hawkes ($K_C \geq K_A^*(t)$)	$o(\Delta)$	$o(\Delta)$	not identified at order Δ
Theorem 3: One-sided exponential variance jumps	$o(\Delta)$	$\sim \alpha_{T_3}^{\text{call}} \Delta, \alpha_{T_3}^{\text{call}} > 0$	$\rightarrow 0$

The ratio limits in Remark 1 follow directly from the option-price expansions. In the two-sided row, substituting the symmetric strikes in (31) into (18) and (19) allows VIX_t to be factored out of both leading terms and replaces $\log(\text{VIX}_t/K)$ in (20) by $\log(\mathbb{k})$ for the put and $-\log(\mathbb{k})$ for the call. The common factor $e^{-r\Delta} \lambda_t \Delta \text{VIX}_t / \Phi(z)$ therefore cancels from the ratio of leading terms. At fixed proportional moneyness, the limiting ratio depends only on $\mathbb{k}$ and the jump-mark parameters, not on λ_t or the VIX level. In the reachable-call one-sided rows, the denominator has a strictly positive order- Δ coefficient while the numerator is $o(\Delta)$, so the ratio converges to zero. If the fixed-increment call is not one-jump reachable, both legs are $o(\Delta)$ and the order results alone do not determine the ratio.

For a given VIX option expiration cycle, the DTE buckets are observed on different quote dates rather than contemporaneously. We therefore estimate a pooled within-cycle log-DTE profile,

$$\log(\mathcal{D}_{t,m,j}) = \psi \log(\Delta_m) + \gamma_{\text{vix}} \log(\text{VIX}_t) + \mathfrak{v}_j + \mathfrak{v}_{\text{cycle}} + \epsilon_{t,m,j}, \quad (33)$$

where t is the quote date associated with DTE bucket m , Δ_m is the numerical DTE measure used for that bucket, and $\mathfrak{v}_j$ and $\mathfrak{v}_{\text{cycle}}$ are symmetric-log-moneyness-bucket and expiration-cycle fixed effects. The quote-date DTE measure Δ_m is distinct from the quote-close-to-cash-settlement horizon Δ in the theorems.

We use five symmetric-log-moneyness buckets labeled 1%, 2.5%, 5%, 10%, and 20%; the exact put and call strikes are determined by (31), so the labels should not be read as equal arithmetic percentage distances on the two sides. We stack observations across the five moneyness buckets and the DTE buckets $m \in \{1, 4, 5, 6, 7\}$. The dependent variable is defined only when both fitted option prices are strictly positive; the quote filters and price construction appear in the note to Table 2.

The coefficient ψ summarizes the common within-expiration-cycle log-DTE profile after conditioning on the VIX level and the two sets of fixed effects. In the standard weekly cycle, DTE buckets correspond to particular quote weekdays, so this gradient reflects their joint within-cycle variation at the observed maturities.

Table 2 reports nested DTE sets. The six point estimates of ψ range from -0.007 to 0.008 ; in the specifications that include $\log(\text{VIX}_t)$, they range from -0.004 to 0.008 . The VIX-level point estimates range from 0.021 to 0.028 , showing a small positive within-cycle association at the observed maturities even though the leading two-sided ratio cancels the VIX level.

The near-flat point-estimate profile across the observed maturities is consistent with equal-order put and call pricing on the two-sided model’s one-jump-reachable set.³ Appendix F complements this option-price evidence with realized-return tail estimates and signed high-frequency jump diagnostics for nearest-maturity VIX futures.

5. Conclusion

We develop a risk-neutral state-dependent marked Hawkes-type mechanism in which signed S&P 500 jumps generate two-sided VIX jumps. Combining an inverse stock–VIX log-jump relation with a local linear-in-intensity squared-VIX approximation determines an intensity update that depends jointly on the pre-jump state and the contemporaneous stock-jump mark. Negative stock jumps generate excitation and

³Our focus on one- to seven-day VIX option quote-date buckets and short-horizon strike-crossing diagnostics distinguishes the empirical analysis from prior VIX option and volatility-jump studies. Related work on self-exciting market dynamics and volatility jumps include Kou (2002), Aït-Sahalia, Cacho-Diaz, and Laeven (2015), Carr and Wu (2017), Bakshi, Crosby, and Gao (2022), Yuan (2022), and Bacon, Bégin, and Gauthier (2024). Park (2016) specifies upward and downward jumps directly in log VIX. Here both VIX jump directions are generated by signed stock jumps through the state-dependent intensity update in (13).

upward VIX jumps, whereas positive stock jumps generate de-excitation and downward VIX jumps. Multiplicative VIX jumps preserve a positive VIX level, and a local admissible stopped construction maintains a nonnegative stock-jump intensity. Thus, a single marked stock-jump process produces both VIX jump directions without introducing an independent VIX jump process.

Our explicit short-maturity option-price expansions establish an order-of-decay separation between the two-sided construction and the two benchmarks allowing only upward VIX jumps. Under the respective local conditions and with positive current jump intensities, every fixed-strike out-of-the-money (OTM) call and every one-jump-reachable OTM put in the two-sided construction have strictly positive leading terms proportional to the remaining option maturity Δ . The two one-sided benchmarks instead yield $o(\Delta)$ put values, while one-jump-reachable calls retain positive order- Δ values. For symmetric absolute log-moneyness relative to current VIX, the put–call ratio therefore converges to a finite, strictly positive limit on the two-sided model’s one-jump-reachable put set. In contrast, it converges to zero under a one-sided benchmark whenever the call has a positive order- Δ coefficient.

The symmetric-strike put–call ratio provides an observable diagnostic of this pricing distinction. In the two-sided model, its limiting value depends on moneyness and the jump-mark parameters, with the current intensity and VIX level canceling from the leading-order expression. The nearly flat within-expiration-cycle log-DTE profile in the option data is consistent with equal-order short-end put and call pricing. The framework thus connects a positivity-preserving stock–VIX jump mechanism to an explicit first-order contribution of downward VIX jumps to short-maturity VIX put value.

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Table 1

Short-maturity VIX option strike crossings, prices, and trading activity

This table reports, by quote-date and option bucket: (i) # crossing, the number of the 514 expiration cycles in which the option has a positive cash-settlement payoff; (ii) the corresponding payoff-positive frequency, equal to # crossing divided by 514; (iii) Black-implied volatility; (iv) the reported offer price; (v) average trading volume (contracts traded); and (vi) average open interest (contracts outstanding). All reported option prices are taken directly from the data, without extrapolation or interpolation. $VIX_{t+\Delta}$ denotes the VIX Special Opening Quotation, disseminated under ticker VRO, computed from opening prices of S&P 500 index options, and used to cash settle expiring VIX options. For the standard weekly cycle, Tuesday is ordinarily the last trading day and settlement ordinarily occurs Wednesday morning. The days-to-expiration (DTE) labels in the panel headings describe the quote-date convention; the crossing payoff in every panel is computed using VRO, so Δ refers to the interval from the quote close to cash settlement rather than to the last traded quote. Option delta is computed using the Black formula; both delta and log-moneyness are verified against OptionMetrics. The sample covers 514 weekly VIX option expiration cycles from October 28, 2015, through August 27, 2025. The 6DTE quote-date bucket is included in the regression analysis in Table 2, but its descriptive crossing statistics are omitted here for brevity. ATM denotes the option whose moneyness is closest to the at-the-money level.

	<u>Panel A: 1DTE quote-date bucket</u> (quote at Monday close)						<u>Panel B: 4DTE quote-date bucket</u> (quote at Friday close)					
	<u>VIX puts</u>			<u>VIX calls</u>			<u>VIX puts</u>			<u>VIX calls</u>		
	-10	-25	-37	39	28	11	-11	-23	-40	43	26	13
Option delta	-10	-5	ATM	ATM	5	10	-12.5	-7.5	ATM	ATM	7.5	12.5
Option moneyness	-10	-5	ATM	ATM	5	10	-12.5	-7.5	ATM	ATM	7.5	12.5
# crossing	53	138	233	150	97	39	63	136	280	142	88	38
{Prob(payoff>0)} (in percent)	{10}	{27}	{45}	{29}	{19}	{8}	{12}	{26}	{54}	{28}	{17}	{7}
Implied volatility (IV, %)	109	105	109	130	137	157	88	89	95	110	122	134
Offer price (\$)	0.14	0.33	0.53	0.67	0.52	0.36	0.21	0.40	0.80	0.94	0.68	0.49
Volume	3,010	6,903	8,263	6,446	6,279	6,585	3,368	6,412	7,669	5,270	5,026	4,662
Open interest	22,731	26,721	26,544	15,955	17,331	23,061	21,795	27,016	29,167	16,566	20,627	24,662
	<u>Panel C: 5DTE quote-date bucket</u> (quote at Thursday close)						<u>Panel D: 7DTE quote-date bucket</u> (quote at Tuesday close)					
	<u>VIX puts</u>			<u>VIX calls</u>			<u>VIX puts</u>			<u>VIX calls</u>		
	-10	-25	-41	45	23	13	-10	-22	-41	46	21	13
Option delta	-15	-7.5	ATM	ATM	10	15	-17.5	-10	ATM	ATM	12.5	17.5
Option moneyness	-15	-7.5	ATM	ATM	10	15	-17.5	-10	ATM	ATM	12.5	17.5
# crossing	65	174	282	134	68	39	50	159	298	142	68	43
{Prob(payoff>0)} (in percent)	{13}	{34}	{55}	{26}	{13}	{8}	{10}	{31}	{58}	{28}	{13}	{8}
Implied volatility (IV, %)	89	90	99	112	127	138	88	90	102	115	133	142
Offer price (\$)	0.19	0.48	0.93	1.05	0.67	0.53	0.22	0.48	1.13	1.30	0.79	0.64
Volume	2,821	6,071	6,397	4,320	4,803	3,706	2,511	4,605	5,329	3,553	3,323	3,189
Open interest	19,920	27,834	28,568	16,282	22,836	25,398	20,941	27,404	29,433	17,507	23,837	25,300

Table 2

Panel regressions for symmetric-strike VIX put–call ratios

We report panel regressions of the log symmetric-strike put–call ratio from equation (32), using the strike construction in equation (31). The estimating specification is equation (33). Observations are indexed by expiration cycle, days-to-expiration bucket, and symmetric moneyness bucket. Columns differ by maturity set; Columns I impose $\gamma_{\text{vix}} \equiv 0$, while Columns II estimate γ_{vix} . All specifications include moneyness and expiration-cycle fixed effects. Standard errors are two-way clustered by expiration cycle and moneyness bucket; t -statistics are reported in parentheses. The second clustering dimension contains only five moneyness buckets; the reported t -statistics are not used for conventional significance claims. In constructing $\mathcal{D}(\mathbb{k}, \Delta)$ in equation (32), we discard VIX option observations for either leg with zero or missing quotes, zero trading volume, or crossed markets (i.e., bid prices exceeding ask prices). The reported results are based on bid–ask midpoint quotes and cubic-spline-fitted prices. A log ratio is retained only when both fitted option prices are strictly positive. Observation counts refer to fitted put–call ratios, not to the number of raw quotes entering the splines. The VIX option sample spans from October 28, 2015, to August 27, 2025. The DTE label follows the market quote-date convention for the expiring weekly VIX option. These DTE labels are distinct from the quote-close-to-cash-settlement horizon in the option-price theorems.

	1DTE, 4DTE, 5DTE, 6DTE, and 7DTE		1DTE, 4DTE, 5DTE, and 6DTE		1DTE, 4DTE, and 5DTE	
	I	II	I	II	I	II
ψ (t -stat.)	0.006 (0.05)	0.008 (0.08)	0.002 (0.02)	0.005 (0.05)	-0.007 (-0.07)	-0.004 (-0.04)
γ_{vix} (t -stat.)	$\equiv 0$	0.028 (2.18)	$\equiv 0$	0.024 (1.86)	$\equiv 0$	0.021 (1.45)
Moneyness fixed effects	Yes	Yes	Yes	Yes	Yes	Yes
Expiration cycle fixed effects	Yes	Yes	Yes	Yes	Yes	Yes
Number of observations	12,850	12,850	10,280	10,280	7,710	7,710

Appendix

A. Mean-field derivation of the linear-in-intensity approximation for VIX_t^2 in (9)

A.1. Setup and the non-closed mean equation

This appendix derives the local squared-VIX approximation and its fixed-horizon error under an explicit stock-model continuation. Theorem 1 retains fixed positive coefficients (b_0, b_λ) and the stopped proxy in (17).

For the log-contract calculation, start under Condition C-2 of Theorem 1 and retain (6) and (13) through τ_{exit} , including any exit-causing jump. Thereafter freeze the intensity at its nonnegative exit value, but let stock jumps continue at that rate with law F_J , without further state updates. Thus $\lambda_u = \lambda_u^{\text{stop}}$ for $u \geq t$, whereas N continues after exit.⁴

Under this continuation and the true-martingale and log-contract assumptions of Section 3.1, equation (8) gives

$$\underbrace{VIX_t^2}_{\text{decimal units}} = V + \frac{2}{\Delta^*} \times \underbrace{\mathbb{E}^{\mathbb{Q}}[e^{J^s} - 1 - J^s]}_{\text{jump wedge}} \times \underbrace{\mathbb{E}_t^{\mathbb{Q}}[\int_t^{t+\Delta^*} \lambda_{\ell-} d\ell]}_{\text{expected integrated intensity}}. \quad (\text{A2})$$

Because, conditional on a jump at time ℓ , the mark law is $F_J(dj)$ and does not depend on $\mathcal{F}_{\ell-}$, the jump contribution to the log-contract expectation separates as

$$\mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{t+\Delta^*} \int (e^j - 1 - j) \mathcal{N}(d\ell, dj) \right] = \Pi \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{t+\Delta^*} \lambda_{\ell-} d\ell \right].$$

For $u \in \mathbb{R}$, the upper-truncated normal mark law gives the moment generating function

$$\mathbb{E}^{\mathbb{Q}}[e^{uJ^s}] = \exp \left(um_J + \frac{u^2 \sigma_J^2}{2} \right) \frac{\Phi(z - \sigma_J u)}{\Phi(z)}, \quad z := \frac{J_{\max} - m_J}{\sigma_J}. \quad (\text{A3})$$

⁴ The continued marked compensator is $\lambda_{\ell-} F_J(dj) d\ell$; the stopped count instead has active intensity $\mathbf{1}_{\{\ell \leq \tau_{\text{exit}}\}} \lambda_{\ell-}$. The pre-exit rate is bounded by λ_{cap} , and the frozen exit rate is finite almost surely, so the continued count is nonexplosive. For every fixed finite $H^\dagger > 0$, compensation and (A3) give

$$\mathbb{E}_t^{\mathbb{Q}} \left[\sup_{t \leq u \leq t+H^\dagger} \lambda_u^2 \right] \leq \lambda_{\text{cap}}^2 + \lambda_{\text{cap}} H^\dagger \left(\lambda_{\text{cap}} + \frac{b_0}{b_\lambda} \right)^2 \mathbb{E}^{\mathbb{Q}}[e^{-4vJ^s}] < \infty. \quad (\text{A1})$$

Before exit the state is bounded, and only the included exit jump can produce an unbounded overshoot. The bound supplies the first and second moments used below.

Therefore, we can determine

$$\Pi := \mathbb{E}^{\mathbb{Q}}[e^{J^s} - 1 - J^s] \quad \text{and} \quad \eta := \mathbb{E}^{\mathbb{Q}}[e^{-2\nu J^s}] = \exp(-2\nu m_J + 2\sigma_J^2 \nu^2) \frac{\Phi(z + 2\nu \sigma_J)}{\Phi(z)}, \quad (\text{A4})$$

$$\Pi = \exp\left(m_J + \frac{\sigma_J^2}{2}\right) \frac{\Phi(z - \sigma_J)}{\Phi(z)} - 1 - \left(m_J - \sigma_J \frac{\phi(z)}{\Phi(z)}\right). \quad (\text{A5})$$

Whenever $\mathbb{E}^{\mathbb{Q}}[e^{J^s}] < \infty$ and $\mathbb{E}^{\mathbb{Q}}[|J^s|] < \infty$, we have $\Pi \geq 0$ (using convexity). That is, since $e^x - 1 - x \geq 0$ for all x , $\Pi \geq 0$ whenever the expectation is finite. Under the maintained continuous truncated-normal mark law with $\sigma_J > 0$, $\text{Prob}(J^s \neq 0) > 0$, hence $\Pi > 0$.

With the coefficients and boundaries fixed, the continuation is time-homogeneous and determined by the current intensity. Write

$$G(\lambda_t) := \mathbb{E}_t^{\mathbb{Q}} \int_t^{t+\Delta^*} \lambda_{\ell-} d\ell. \quad (\text{A6})$$

For an initial state $\lambda \geq 0$ outside $(\lambda_{\text{exit}}, \lambda_{\text{cap}})$, take the intensity to be absorbing, so $G(\lambda) = \lambda \Delta^*$. For a pre-exit state λ , integration with respect to the conditional mark kernel gives

$$\int A(\lambda, j) F_J(dj) = \frac{b_0 + b_\lambda \lambda}{b_\lambda} \left(\mathbb{E}^{\mathbb{Q}}[e^{-2\nu J^s}] - 1 \right) = \frac{b_0(\eta - 1)}{b_\lambda} + (\eta - 1)\lambda. \quad (\text{A7})$$

Therefore,

$$\lambda_{\ell-} \int A(\lambda_{\ell-}, j) F_J(dj) = (\eta - 1) \frac{b_0}{b_\lambda} \lambda_{\ell-} + (\eta - 1) \lambda_{\ell-}^2. \quad (\text{A8})$$

Define the exact conditional mean

$$m(\ell) := \mathbb{E}_t^{\mathbb{Q}}[\lambda_\ell], \quad \ell \geq t, \quad m(t) = \lambda_t. \quad (\text{A9})$$

The continued intensity state obeys

$$d\lambda_\ell = \mathbf{1}_{\{\ell \leq \tau_{\text{exit}}\}} \left\{ \beta(\mu - \lambda_{\ell-}) d\ell + \int_{(-\infty, J_{\text{max}}]} A(\lambda_{\ell-}, j) \mathcal{N}(d\ell, dj) \right\}, \quad (\text{A10})$$

$$\mathbb{E}^{\mathbb{Q}}[\mathcal{N}(d\ell, dj) \mid \mathcal{F}_{\ell-}] = \lambda_{\ell-} F_J(dj) d\ell.$$

The predictable indicator includes the exit-causing jump. Compensation and the bound (A1) in footnote 4 give, for almost every $\ell \geq t$,

$$\frac{d}{d\ell} \mathbb{E}_t^{\mathbb{Q}}[\lambda_\ell] = \mathbb{E}_t^{\mathbb{Q}} \left[\mathbf{1}_{\{\ell < \tau_{\text{exit}}\}} \left\{ \beta(\mu - \lambda_\ell) + \lambda_\ell \int A(\lambda_\ell, j) F_J(dj) \right\} \right]. \quad (\text{A11})$$

Substituting (A7) and expanding the indicator in (A11) yields the exact non-closed mean equation:

$$m'(\ell) = \beta(\mu - m(\ell)) + (\eta - 1) \left(\frac{b_0}{b_\lambda} m(\ell) + \mathbb{E}_t^\mathbb{Q}[\lambda_\ell^2] \right) - \mathbb{E}_t^\mathbb{Q} \left[\mathbf{1}_{\{\tau_{\text{exit}} \leq \ell\}} \left\{ \beta(\mu - \lambda_\ell) + (\eta - 1) \left(\frac{b_0}{b_\lambda} \lambda_\ell + \lambda_\ell^2 \right) \right\} \right]. \quad (\text{A12})$$

The last term removes the mean drift after the state has frozen.

A.2. Local linearization around a reference level $\bar{\lambda}$

The constants have distinct roles: Π is the log-contract wedge, while $\eta - 1$ governs the mean intensity update in (A7). We use a mean-field moment closure that replaces the conditional second moment by the squared conditional mean:

$$\mathbb{E}_t^\mathbb{Q}[\lambda_\ell^2] = m(\ell)^2 + \text{Var}_t(\lambda_\ell) \approx m(\ell)^2. \quad (\text{A13})$$

Dropping the conditional variance in (A13) and the exit correction in (A12) gives the Riccati approximation

$$m'(\ell) \approx \beta(\mu - m(\ell)) + (\eta - 1) \left(\frac{b_0}{b_\lambda} m(\ell) + m(\ell)^2 \right), \quad m(t) = \lambda_t. \quad (\text{A14})$$

Define the mean-field drift

$$f(m) = \beta(\mu - m) + (\eta - 1) \left(\frac{b_0}{b_\lambda} m + m^2 \right). \quad (\text{A15})$$

Then

$$f'(m) = -\beta + (\eta - 1) \left(\frac{b_0}{b_\lambda} + 2m \right). \quad (\text{A16})$$

Choose $\bar{\lambda} \geq 0$ with $f(\bar{\lambda}) = 0$ and impose local stability:

$$\tilde{\beta} := -f'(\bar{\lambda}) = \beta - (\eta - 1) \left(\frac{b_0}{b_\lambda} + 2\bar{\lambda} \right) > 0. \quad (\text{A17})$$

For m near $\bar{\lambda}$, a first-order Taylor expansion gives

$$m'(\ell) \approx f(\bar{\lambda}) + f'(\bar{\lambda})(m(\ell) - \bar{\lambda}). \quad (\text{A18})$$

Because f is quadratic, (A12) and (A17) give the exact identity

$$m'(\ell) = -\tilde{\beta}(m(\ell) - \bar{\lambda}) + (\eta - 1) \mathbb{E}_t^\mathbb{Q}[(\lambda_\ell - \bar{\lambda})^2] - \mathbb{E}_t^\mathbb{Q}[\mathbf{1}_{\{\tau_{\text{exit}} \leq \ell\}} f(\lambda_\ell)]. \quad (\text{A19})$$

Variation of constants therefore gives

$$m(\ell) = \bar{\lambda} + (\lambda_t - \bar{\lambda}) e^{-\tilde{\beta}(\ell-t)} + R_\ell, \quad (\text{A20})$$

where

$$R_\ell = \int_t^\ell e^{-\tilde{\beta}(\ell-u)} \left\{ (\eta - 1) \mathbb{E}_t^\mathbb{Q}[(\lambda_u - \bar{\lambda})^2] - \mathbb{E}_t^\mathbb{Q}[\mathbf{1}_{\{\tau_{\text{exit}} \leq u\}} f(\lambda_u)] \right\} du. \quad (\text{A21})$$

Indeed,

$$\mathbb{E}_t^\mathbb{Q}[(\lambda_u - \bar{\lambda})^2] = \text{Var}_t(\lambda_u) + (m(u) - \bar{\lambda})^2. \quad (\text{A22})$$

Thus R_ℓ is an exact remainder containing the variance, linearization, and exit corrections. Dropping R_ℓ and integrating (A20) over $[t, t + \Delta^*]$ gives

$$\mathbb{E}_t^\mathbb{Q} \int_t^{t+\Delta^*} \lambda_u du \approx \bar{\lambda} \Delta^* + (\lambda_t - \bar{\lambda}) \frac{1 - e^{-\tilde{\beta} \Delta^*}}{\tilde{\beta}}. \quad (\text{A23})$$

Equation (A23) becomes the linear-in- λ_t approximation:

$$\mathbb{E}_t^\mathbb{Q} \int_t^{t+\Delta^*} \lambda_u du \approx \bar{\lambda}(\Delta^* - \psi(\Delta^*)) + \psi(\Delta^*) \lambda_t, \quad \text{where} \quad \psi(\Delta^*) := \frac{1 - e^{-\tilde{\beta} \Delta^*}}{\tilde{\beta}}. \quad (\text{A24})$$

Finally, substituting (A24) into (A2) yields a local approximation for VIX_t^2 :

$$\text{VIX}_t^2 \approx b_0 + b_\lambda \lambda_t, \quad \text{where} \quad b_\lambda = \frac{2\Pi}{\Delta^*} \psi(\Delta^*) \quad \text{and} \quad b_0 = V + \frac{2\Pi}{\Delta^*} \bar{\lambda}(\Delta^* - \psi(\Delta^*)). \quad (\text{A25})$$

Equations (A17), (A25), and (A29) determine the coefficients and reference pair jointly. Since $\Pi > 0$ and $0 < \psi(\Delta^*) < \Delta^*$ when $\tilde{\beta} > 0$, (A25) gives $b_\lambda > 0$ and $b_0 \geq V > 0$ for $\bar{\lambda} \geq 0$.

Fixed-horizon approximation error. Extend the function in (A24) to $\psi(s) = (1 - e^{-\tilde{\beta}s})/\tilde{\beta}$ for $s \geq 0$. Retaining R_ℓ in (A20), integrating, and using (A2) and (A25) gives

$$\begin{aligned} \text{VIX}_t^2 - (b_0 + b_\lambda \lambda_t) &= \frac{2\Pi}{\Delta^*} \int_t^{t+\Delta^*} R_\ell d\ell \\ &= \frac{2\Pi}{\Delta^*} \int_t^{t+\Delta^*} \psi(t + \Delta^* - u) \left\{ (\eta - 1) \mathbb{E}_t^\mathbb{Q}[(\lambda_u - \bar{\lambda})^2] \right. \\ &\quad \left. - \mathbb{E}_t^\mathbb{Q}[\mathbf{1}_{\{\tau_{\text{exit}} \leq u\}} f(\lambda_u)] \right\} du. \end{aligned} \quad (\text{A26})$$

The moment bound (A1) (in footnote 4) and the quadratic form of f justify the interchange of integrals.

Since $\psi \geq 0$ and $\int_0^{\Delta^*} \psi(s) ds = (\Delta^* - \psi(\Delta^*))/\tilde{\beta}$, the error satisfies

$$\begin{aligned} |\text{VIX}_t^2 - (b_0 + b_\lambda \lambda_t)| &\leq \frac{2\Pi\{\Delta^* - \psi(\Delta^*)\}}{\Delta^* \tilde{\beta}} \left\{ |\eta - 1| \sup_{t \leq u \leq t+\Delta^*} \mathbb{E}_t^\mathbb{Q}[(\lambda_u - \bar{\lambda})^2] \right. \\ &\quad \left. + \sup_{t \leq u \leq t+\Delta^*} \mathbb{E}_t^\mathbb{Q}[\mathbf{1}_{\{\tau_{\text{exit}} \leq u\}} |f(\lambda_u)|] \right\}. \end{aligned} \quad (\text{A27})$$

The bound separates conditional mean-square deviation from $\bar{\lambda}$ and the exit correction. At $\eta = 1$, only the exit correction remains. It assesses accuracy over the fixed VIX horizon $\Delta^* = 30/365$, independently of the option-maturity limit $\Delta \downarrow 0$.

We call

$$\mathcal{S} := (\bar{\lambda}, \tilde{\beta}, b_0, b_\lambda) \quad (\text{A28})$$

a selected mean-field solution if it satisfies (A17), the coefficient formulas in (A25), the fixed-point restriction

$$0 = \beta(\mu - \bar{\lambda}) + (\eta - 1) \left(\frac{b_0}{b_\lambda} \bar{\lambda} + \bar{\lambda}^2 \right), \quad (\text{A29})$$

and the positivity restrictions

$$\bar{\lambda} \geq 0, \quad \tilde{\beta} > 0, \quad b_0 > 0, \quad b_\lambda > 0. \quad (\text{A30})$$

All calculations are conditional on a selected $\mathcal{S}$.

Local solvability of the mean-field system. With the stock-mark law and remaining parameters fixed, the self-consistency system admits a locally unique branch of selected solutions for $\mu > 0$ and sufficiently small positive v . To see this, regard η as a continuation parameter and substitute (A25) into (A17) and (A29). The latter two restrictions are equivalently

$$0 = \tilde{\beta} - \beta + (\eta - 1) \left(\frac{b_0}{b_\lambda} + 2\bar{\lambda} \right), \quad (\text{A31})$$

$$0 = \tilde{\beta} \bar{\lambda} + (\eta - 1) \bar{\lambda}^2 - \beta \mu, \quad (\text{A32})$$

where $\frac{b_0}{b_\lambda}$ is the smooth function of $(\tilde{\beta}, \bar{\lambda})$ given by (A25) for $\tilde{\beta} > 0$. At $\eta = 1$, (A31)–(A32) have the solution $(\tilde{\beta}, \bar{\lambda}) = (\beta, \mu)$. The Jacobian of their left-hand sides with respect to $(\tilde{\beta}, \bar{\lambda})$ at this solution is

$$\begin{pmatrix} 1 & 0 \\ \mu & \beta \end{pmatrix}, \quad (\text{A33})$$

whose determinant is $\beta > 0$. The implicit-function theorem therefore gives a unique nearby solution for η sufficiently close to one. As $v \downarrow 0$, (A4) gives $\eta \rightarrow 1$, so continuity preserves (A30), with $\bar{\lambda} > 0$, for all

sufficiently small positive v .⁵

Implications for the one-jump option coefficients. Under the specified continuation, the log-contract VIX mapping is

$$\mathcal{V}(\lambda) := \sqrt{V + \frac{2\Pi}{\Delta^*}G(\lambda)}, \quad \lambda \geq 0. \quad (\text{A34})$$

At an absorbing state, $\mathcal{V}(\lambda) = \sqrt{V + 2\Pi\lambda}$ by (A6). For every $\lambda \geq 0$, nonnegativity yields

$$\left| \mathcal{V}(\lambda) - \sqrt{b_0 + b_\lambda \lambda} \right| \leq \frac{\left| V + \frac{2\Pi}{\Delta^*}G(\lambda) - (b_0 + b_\lambda \lambda) \right|}{\sqrt{V} + \sqrt{b_0}}. \quad (\text{A35})$$

The bound (A1), uniformly over initial states in $(\lambda_{\text{exit}}, \lambda_{\text{cap}})$, and the absorbing-state formula imply $\sup_{\lambda \geq 0} \mathcal{V}(\lambda)/(1 + \sqrt{\lambda}) < \infty$. Moreover, G is continuous on this open interval.⁶

Fix $g_K(y) = (y - K)^+$ for a call or $g_K(y) = (K - y)^+$ for a put, with K strictly OTM under both $\mathcal{V}(\lambda_t)$ and Y_t . The growth bound and (A1) give a uniform fourth payoff moment. Continuity makes the no-jump payoff zero for all sufficiently small Δ and makes the terminal one-jump payoff converge to the immediate post-jump payoff for F_J -almost every mark. By (13), these one-jump payoffs are bounded by a constant multiple of $1 + e^{-vj}$, which is integrable by (A3). The stopped-count argument in Appendix C therefore shows that the price based on $g_K(\mathcal{V}(\lambda_{t+\Delta}))$ has order- Δ coefficient $\lambda_t \int g_K(\mathcal{V}(\lambda_t + A(\lambda_t, j))) F_J(dj)$. The proxy coefficient instead uses $Y_t e^{-vj}$, by (13). Since g_K is one-Lipschitz,

$$\begin{aligned} & \left| \lambda_t \int g_K(\mathcal{V}(\lambda_t + A(\lambda_t, j))) F_J(dj) - \lambda_t \int g_K(Y_t e^{-vj}) F_J(dj) \right| \\ & \leq \lambda_t \int \left| \mathcal{V}(\lambda_t + A(\lambda_t, j)) - Y_t e^{-vj} \right| F_J(dj). \end{aligned} \quad (\text{A36})$$

All mark integrals are over $(-\infty, J_{\max}]$ and include exit-causing jumps.

For calls, (13) gives

⁵Theorem 1 additionally imposes Condition C-2, and at fixed symmetric moneyness $\mathbb{k} > 1$, Remark 1 requires $vJ_{\max} > \log \mathbb{k}$.

⁶To see this, couple nearby stopped intensity states using a common rate- λ_{cap} Poisson clock with independent uniform thinning variables and marks of law F_J . The flow and jump update are continuous, and any finite continuous exit is a transverse crossing. Almost surely, no thinning variable equals its acceptance threshold and no jump lands exactly on a boundary. With finitely many candidate arrivals on the fixed horizon, the integrated intensities therefore converge almost surely; (A1) gives uniform integrability and convergence of their expectations. The absorbing-state formula gives continuity outside the interval, away from its two boundaries. Since $j \mapsto \lambda_t + A(\lambda_t, j)$ is strictly decreasing, $\mathcal{V}$ is continuous at λ_t and at the post-jump state for F_J -almost every mark.

$$\lambda_t + A(\lambda_t, j) = \left(\lambda_t + \frac{b_0}{b_\lambda} \right) e^{-2vj} - \frac{b_0}{b_\lambda} \longrightarrow \infty \quad \text{as } j \rightarrow -\infty.$$

These states eventually exceed λ_{cap} , where the absorbing-state formula implies $\mathcal{V}(\lambda_t + A(\lambda_t, j)) \rightarrow \infty$. Since $F_J((-\infty, j)) > 0$ for every finite $j < J_{\text{max}}$, the exact one-jump call coefficient is strictly positive for every fixed OTM call strike.

With $\lambda_t > 0$ and $g_K(y) = (K - y)^+$, a sufficient condition for a strictly positive exact-mapping put coefficient is

$$\int |\mathcal{V}(\lambda_t + A(\lambda_t, j)) - Y_t e^{-vj}| F_J(dj) < \int g_K(Y_t e^{-vj}) F_J(dj). \quad (\text{A37})$$

When (A37) holds for the put leg, both exact-mapping option prices have strictly positive leading terms proportional to Δ . ■

B. Proof of Lemma 1: Admissibility under the local VIX-proxy update

We prove the jump-time assertion. Let τ be a jump time of N and write

$$\lambda_- := \lambda_{\tau-}, \quad j := J_\tau^s, \quad b := \frac{b_0}{b_\lambda} > 0. \quad (\text{B1})$$

Since $b_\lambda > 0$, (13) gives

$$A(\lambda_-, j) = \frac{b_0 + b_\lambda \lambda_-}{b_\lambda} (e^{-2vj} - 1) = (\lambda_- + b)(e^{-2vj} - 1). \quad (\text{B2})$$

Therefore the post-jump intensity is

$$\lambda_\tau = \lambda_- + A(\lambda_-, j) = (\lambda_- + b)e^{-2vj} - b. \quad (\text{B3})$$

This is the exact jump-time identity implied by local proxy consistency.

Since $b > 0$, for any nonnegative pre-jump intensity λ_- ,

$$\begin{aligned} \lambda_\tau \geq 0 &\iff (\lambda_- + b)e^{-2vj} \geq b \\ &\iff \lambda_- + b \geq b e^{2vj} \\ &\iff \lambda_- \geq b(e^{2vj} - 1). \end{aligned} \quad (\text{B4})$$

Thus (B4) is the mark-by-mark admissibility condition under the maintained restriction $b = \frac{b_0}{b_\lambda} > 0$.

Now impose the upper support restriction $J_\tau^s \leq J_{\max}$ almost surely and assume $J_{\max} > 0$. Since $v > 0$ and $b > 0$, the function $j \mapsto b(e^{2vj} - 1)$ is nondecreasing on $\mathbb{R}$. Consequently, for every realized jump mark $j \leq J_{\max}$,

$$b(e^{2vj} - 1) \leq b(e^{2vJ_{\max}} - 1). \quad (\text{B5})$$

Hence the maintained pre-jump lower bound

$$\lambda_{\tau-} \geq \frac{b_0}{b_\lambda} (e^{2vJ_{\max}} - 1) = b(e^{2vJ_{\max}} - 1) = \lambda_{\text{exit}} \quad (\text{B6})$$

implies the exact condition (B4) for every admissible realization of J_τ^s . Therefore $\lambda_\tau \geq 0$ at that jump time. This proves Lemma 1. The bound in (B6) is a pre-jump admissibility condition, not an invariant lower boundary: either the deterministic flow or an admissible de-excitation jump may move the state below the buffer. The stopped construction in (17) records the exit state and then holds it fixed.

For completeness, note how the jump-time conclusion interacts with the between-jump drift. Between two successive jump times, the intensity solves

$$d\lambda_u = \beta(\mu - \lambda_u) du, \quad \beta > 0, \quad (\text{B7})$$

and hence, starting from any time t_0 with no intervening jump,

$$\lambda_u = \mu + (\lambda_{t_0} - \mu) e^{-\beta(u-t_0)} = e^{-\beta(u-t_0)} \lambda_{t_0} + (1 - e^{-\beta(u-t_0)}) \mu, \quad u \geq t_0. \quad (\text{B8})$$

For the stopped construction in (17), every jump applied before or at exit has a pre-jump state satisfying (B6), so its post-jump state is nonnegative. A continuous exit occurs at one of the positive localization boundaries. Since the exit state is subsequently held fixed, the stopped state remains nonnegative and the stopped VIX proxy remains strictly positive.

Throughout this Appendix, $b = \frac{b_0}{b_\lambda} > 0$ is maintained. Under this maintained coefficient restriction, the upper bound on positive stock-jump marks is essential for a finite uniform pre-jump lower bound of the form (B6). If the conditional mark law had an unbounded right tail, then for every finite λ_- there would

be admissible marks j with $\lambda_- < b(e^{2v}j - 1)$, and (B3) would give $\lambda_\tau < 0$. The upper-truncated mark distribution rules out exactly these arbitrarily large positive stock-jump marks. ■

C. Proof of Theorem 1

Throughout this proof, all terminal payoffs are evaluated under the stopped construction in (17). By the two-sided state margin in C-2, the no-jump deterministic path remains in $(\lambda_{\text{exit}}, \lambda_{\text{cap}})$ on $[t, t + \Delta]$ for all sufficiently small Δ . Before the first jump, $\lambda_{\ell-} = \lambda_t + O(\Delta)$ uniformly in $\ell \in (t, t + \Delta]$, so the pre-jump state is inside the localized region and Lemma 1 makes the jump admissible. If the jump exits through either boundary, the right-continuous stopped process includes the post-jump value and then freezes. Thus the upper localization does not remove large one-jump call payoffs, and the lower localization does not remove the first jump that generates a put payoff.

Under the maintained local proxy, write

$$Y_u = \sqrt{b_0 + b_\lambda \lambda_u}, \quad z = \frac{J_{\max} - m_J}{\sigma_J}, \quad m_{\text{vix}} = -v m_J, \quad \sigma_{\text{vix}} = v \sigma_J. \quad (\text{C1})$$

The stock-jump mark has density

$$F_J(dj) = \frac{1}{\Phi(z)} \frac{1}{\sigma_J} \phi\left(\frac{j - m_J}{\sigma_J}\right) \mathbf{1}_{\{j \leq J_{\max}\}} dj. \quad (\text{C2})$$

At an admitted jump time ℓ with mark j , (10)–(13) give $Y_\ell = Y_{\ell-} e^{-vj}$. For a fixed strike K , define

$$\kappa_K := \log\left(\frac{K}{Y_t}\right), \quad j_K := -\frac{\kappa_K}{v} = \frac{\log(Y_t/K)}{v}, \quad K_{\min} = Y_t e^{-vJ_{\max}}. \quad (\text{C3})$$

Step 1: Stopped first-jump reduction. Let

$$\mathcal{N}(d\ell, dj) = \sum_i \delta_{\{\tau_i, J_i^s\}}(d\ell, dj) \quad (\text{C4})$$

be the marked jump measure. For every Borel set $B \subseteq (-\infty, J_{\max}]$, the stopped count $\mathcal{N}((t, u \wedge \tau_{\text{exit}}] \times B)$ has compensator

$$\int_t^{u \wedge \tau_{\text{exit}}} \lambda_{\ell-} F_J(B) d\ell. \quad (\text{C5})$$

Before exit, the active jump intensity is bounded by λ_{cap} . The standard factorial-moment bound for a point

process with bounded predictable intensity therefore gives

$$\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta}^{\text{stop}} - N_t \geq 2) = O(\Delta^2). \quad (\text{C6})$$

On a path with one admitted jump at ℓ and no later jump, the stopped path follows the deterministic flow

$$\lambda_u = \mu + (\lambda_\ell - \mu)e^{-\beta(u-\ell)}, \quad \ell \leq u \leq (t + \Delta) \wedge \tau_{\text{exit}}, \quad (\text{C7})$$

and freezes if this flow subsequently exits the localized region. Since a non-exiting jump leaves $\lambda_\ell \leq \lambda_{\text{cap}}$

and the map $\lambda \mapsto \sqrt{b_0 + b_\lambda \lambda}$ has derivative bounded above by $b_\lambda / (2\sqrt{b_0})$ on $[0, \infty)$,

$$|Y_{t+\Delta}^{\text{stop}} - Y_\ell| \leq \frac{b_\lambda \beta}{2\sqrt{b_0}} |\lambda_\ell - \mu| \Delta \leq C_{\text{drift}} \Delta. \quad (\text{C8})$$

Here $C_{\text{drift}} := \frac{b_\lambda \beta}{2\sqrt{b_0}} (\lambda_{\text{cap}} + \mu)$ is a finite deterministic constant, independent of Δ , the jump time, and the realized jump mark. If the jump itself triggers stopping, this correction is zero. Since the probability of one stopped jump is $O(\Delta)$, the expected post-jump finite-variation correction is $O(\Delta^2)$.

Before an exit-causing jump, the state is bounded by λ_{cap} ; if the exit is caused by an excitation jump, its included post-jump value is controlled by $e^{-2\nu J^s}$. Condition C-3 therefore gives a uniform fourth conditional moment for $1 + \sup_{t \leq u \leq t+\Delta} Y_u^{\text{stop}}$. Combining this moment bound with (C6), conditional Hölder's inequality gives

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}} \left[\left(1 + \sup_{t \leq u \leq t+\Delta} Y_u^{\text{stop}} \right) \mathbf{1}_{\{N_{t+\Delta}^{\text{stop}} - N_t \geq 2\}} \right] \\ & \leq \left(\mathbb{E}_t^{\mathbb{Q}} \left[\left(1 + \sup_{t \leq u \leq t+\Delta} Y_u^{\text{stop}} \right)^4 \right] \right)^{1/4} \left(\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta}^{\text{stop}} - N_t \geq 2) \right)^{3/4} \\ & = O(\Delta^{3/2}) = o(\Delta). \end{aligned} \quad (\text{C9})$$

The same estimate controls the immediate first-jump payoff on paths with two or more stopped jumps, because the included first-jump value is bounded by the same path supremum. Thus paths with at least two stopped jumps do not affect the leading coefficient. Because the strike is fixed and OTM, the no-jump path produces zero payoff for all sufficiently small Δ .

For a fixed OTM call $K > Y_t$, the one-Lipschitz payoff and the preceding bounds give

$$\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta}^{\text{stop}} - K)^+] = \mathbb{E}_t^{\mathbb{Q}} \left[\int_{(t, (t+\Delta) \wedge \tau_{\text{exit}}]} \int (Y_{\ell-} e^{-\nu j} - K)^+ \mathbf{1}_{\{N_{\ell-}^{\text{stop}} = N_t\}} \mathcal{N}(d\ell, dj) \right] + o(\Delta) \quad (\text{C10})$$

$$= \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{(t+\Delta) \wedge \tau_{\text{exit}}} \lambda_{\ell-} \mathbf{1}_{\{N_{\ell-}^{\text{stop}} = N_t\}} \int (Y_{\ell-} e^{-\nu j} - K)^+ F_J(dj) d\ell \right] + o(\Delta) \quad (\text{C11})$$

$$= \lambda_t \Delta \int (Y_t e^{-\nu j} - K)^+ F_J(dj) + o(\Delta). \quad (\text{C12})$$

The last equality follows by conditional dominated convergence: before exit the state is bounded by λ_{cap} , the mark integral is locally Lipschitz in $Y_{\ell-}$ with at most linear growth, the stopped states are conditionally right-continuous at t , and the two-sided state margin implies $\tau_{\text{exit}} > t$ almost surely. The identical argument for a fixed OTM put $K < Y_t$ gives

$$\mathbb{E}_t^{\mathbb{Q}}[(K - Y_{t+\Delta}^{\text{stop}})^+] = \lambda_t \Delta \int (K - Y_t e^{-\nu j})^+ F_J(dj) + o(\Delta). \quad (\text{C13})$$

Multiplication by $e^{-r\Delta}$ gives the localized option prices.

Step 2: Evaluate the call integral. For $K > Y_t$, $j_K < 0 < J_{\text{max}}$, so the upper truncation does not bind at the crossing threshold. Therefore

$$\int (Y_t e^{-\nu j} - K)^+ F_J(dj) = \frac{1}{\Phi(z)} \int_{-\infty}^{j_K} (Y_t e^{-\nu j} - K) \frac{1}{\sigma_J} \phi\left(\frac{j - m_J}{\sigma_J}\right) dj. \quad (\text{C14})$$

Completing the square yields

$$\int_{-\infty}^{j_K} e^{-\nu j} \frac{1}{\sigma_J} \phi\left(\frac{j - m_J}{\sigma_J}\right) dj = e^{m_{\text{vix}} + \frac{1}{2}\sigma_{\text{vix}}^2} \Phi(d_1), \quad \int_{-\infty}^{j_K} \frac{1}{\sigma_J} \phi\left(\frac{j - m_J}{\sigma_J}\right) dj = \Phi(d_2), \quad (\text{C15})$$

where $d_2 = (j_K - m_J)/\sigma_J$ and $d_1 = d_2 + \sigma_{\text{vix}}$, as in (20). Hence

$$\int (Y_t e^{-\nu j} - K)^+ F_J(dj) = \frac{1}{\Phi(z)} \left[Y_t e^{m_{\text{vix}} + \frac{1}{2}\sigma_{\text{vix}}^2} \Phi(d_1) - K \Phi(d_2) \right]. \quad (\text{C16})$$

The integrand is strictly positive on $(-\infty, j_K)$, which has positive F_J -mass. Substitution into (C12) proves (19) and the strict positivity of its coefficient.

Step 3: Evaluate the put integral. For $K < Y_t$, $j_K > 0$, and a positive one-jump put payoff is possible exactly

when

$$j_K < J_{\max} \iff K > Y_t e^{-\nu J_{\max}} = K_{\min}. \quad (\text{C17})$$

If $0 < K \leq K_{\min}$, then $K - Y_t e^{-\nu j} \leq 0$ for every $j \leq J_{\max}$, so the leading one-jump put integral is zero, including at $K = K_{\min}$. Consequently,

$$\text{Put}_t^{\text{stop}}(K, \Delta) = o(\Delta), \quad 0 < K \leq K_{\min}. \quad (\text{C18})$$

For $K_{\min} < K < Y_t$,

$$\int (K - Y_t e^{-\nu j})^+ F_J(dj) = \frac{1}{\Phi(z)} \int_{j_K}^{J_{\max}} (K - Y_t e^{-\nu j}) \frac{1}{\sigma_J} \phi\left(\frac{j - m_J}{\sigma_J}\right) dj \quad (\text{C19})$$

$$= \frac{1}{\Phi(z)} \left[K \{ \Phi(z) - \Phi(d_2) \} - Y_t e^{m_{\text{vix}} + \frac{1}{2} \sigma_{\text{vix}}^2} \{ \Phi(z + \sigma_{\text{vix}}) - \Phi(d_1) \} \right]. \quad (\text{C20})$$

Here the integrand is strictly positive on $(j_K, J_{\max})$, which has positive F_J -mass. Substitution into the put expansion proves (18) and the strict positivity of its reachable-put coefficient. This completes the proof. ■

D. Theorem 2: Standard linear Hawkes process and VIX option pricing

Fix the conditioning time t and a finite nonnegative current state $0 \leq \lambda_t < \infty$. Consider the fixed-increment linear Hawkes benchmark

$$d\lambda_u = \beta(\mu - \lambda_{u-}) du + A_L dN_u, \quad A_L > 0, \quad \beta > A_L, \quad \mu > 0, \quad (\text{D1})$$

where N is the simple counting process of stock-jump arrivals, λ is the right-continuous intensity state, and λ_{u-} is the predictable arrival intensity. The compensator of $N_u - N_t$, for $u \geq t$, is $\int_t^u \lambda_{\ell-} d\ell$. At an arrival time τ , the count increases by one and the state increases by A_L :

$$N_\tau - N_{\tau-} = 1, \quad \lambda_\tau - \lambda_{\tau-} = A_L > 0. \quad (\text{D2})$$

This intensity increment is independent of the realized stock-jump mark; positivity therefore places no upper truncation requirement on that mark law.

On the event $\{N_{t+s} = N_t\}$, integration of the between-arrival drift gives

$$\lambda_{t+s} = \mu + (\lambda_t - \mu) e^{-\beta s}, \quad s \geq 0. \quad (\text{D3})$$

The condition $A_L/\beta < 1$ is the subcriticality condition for the exponential excitation kernel. The present calculation conditions on λ_t and does not impose stationary initialization. Between arrivals, (D3) is a convex combination of the current state and μ ; together with the positive arrival increments, this preserves nonnegativity. After n new arrivals, the intensity is at most $\max\{\lambda_t, \mu\} + nA_L$.⁷

For this benchmark only, retain the stock dynamics in (1) and the log-contract assumptions of Section 3.1, and take the conditionally i.i.d. stock-jump marks to have the state-independent, untruncated Gaussian law with $\sigma_J > 0$,

$$J^s \sim \text{Normal}(m_J, \sigma_J^2), \quad (\text{D4})$$

and define

$$\Pi = \mathbb{E}^{\mathbb{Q}}[e^{J^s} - 1 - J^s] = \exp\left(m_J + \frac{1}{2}\sigma_J^2\right) - 1 - m_J. \quad (\text{D5})$$

Let

$$H_t(u) := \mathbb{E}_t^{\mathbb{Q}}[\lambda_{t+u}], \quad H_t(0) = \lambda_t. \quad (\text{D6})$$

Compensation gives $\mathbb{E}_t^{\mathbb{Q}}[N_{t+u} - N_t] = \int_0^u H_t(v) dv$. Taking conditional expectations in the integrated form of (D1) and differentiating therefore gives the closed ordinary differential equation

$$\frac{dH_t(u)}{du} = \beta\{\mu - H_t(u)\} + A_L H_t(u) = \beta\mu - (\beta - A_L)H_t(u). \quad (\text{D7})$$

Solving (D7),

$$H_t(u) = \frac{\beta\mu}{\beta - A_L} + \left(\lambda_t - \frac{\beta\mu}{\beta - A_L}\right) e^{-(\beta - A_L)u}. \quad (\text{D8})$$

The between-arrival flow in (D3) relaxes toward μ at rate β , whereas this conditional mean includes future arrivals and relaxes toward $\beta\mu/(\beta - A_L)$ at rate $\beta - A_L$. Hence the expected integrated intensity over the fixed VIX horizon Δ^* is

$$G(\lambda_t) := \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\Delta^*} \lambda_{u-} du\right] = \int_0^{\Delta^*} H_t(u) du \quad (\text{D9})$$

$$= \frac{\beta\mu}{\beta - A_L} \Delta^* + \left(\lambda_t - \frac{\beta\mu}{\beta - A_L}\right) \frac{1 - e^{-(\beta - A_L)\Delta^*}}{\beta - A_L}. \quad (\text{D10})$$

Substituting (D10) into the log-contract VIX mapping gives

⁷This linear bound on the arrival rate ensures nonexplosion and finite conditional moments on bounded time intervals.

$$\text{VIX}_t^2 = V + \frac{2\Pi}{\Delta^*} G(\lambda_t) = w_0 + w_\lambda \lambda_t, \quad (\text{D11})$$

where

$$w_\lambda = 2\Pi \frac{1 - e^{-(\beta - A_L)\Delta^*}}{(\beta - A_L)\Delta^*} \quad \text{and} \quad w_0 = V + \frac{2\beta\mu}{\beta - A_L} \Pi \left(1 - \frac{1 - e^{-(\beta - A_L)\Delta^*}}{(\beta - A_L)\Delta^*} \right). \quad (\text{D12})$$

Equations (D11)–(D12) give the exact benchmark mapping (21). Since $e^x - 1 - x > 0$ for $x \neq 0$ and $\sigma_J > 0$,

we have $\Pi > 0$. Also,

$$0 < \frac{1 - e^{-(\beta - A_L)\Delta^*}}{(\beta - A_L)\Delta^*} < 1. \quad (\text{D13})$$

Thus $w_\lambda > 0$ and $w_0 > V > 0$. The horizon Δ^* and these coefficients are fixed throughout the option-maturity limit.

We now prove the small-maturity asymptotic option expansion. Set

$$Y_u := \text{VIX}_u = \sqrt{w_0 + w_\lambda \lambda_u}, \quad Y_t = \text{VIX}_t, \quad Y_t^{(1)} := \sqrt{w_0 + w_\lambda (\lambda_t + A_L)}. \quad (\text{D14})$$

Here $Y_t^{(1)}$ is the reference VIX level obtained by applying one intensity increment to the conditioning state λ_t . At an actual arrival time τ ,

$$Y_\tau - Y_{\tau-} = \frac{w_\lambda A_L}{\sqrt{w_0 + w_\lambda (\lambda_{\tau-} + A_L)} + \sqrt{w_0 + w_\lambda \lambda_{\tau-}}} > 0. \quad (\text{D15})$$

The intensity increment is fixed, but the induced VIX jump depends on the pre-arrival state. Neither increment depends on the signed stock-jump mark. Fix a positive strike $K \neq Y_t$. All asymptotic bounds below are conditional on $\mathcal{F}_t$, with λ_t , K , and the model parameters held fixed.

First, the no-arrival flow gives the exact first-arrival probability

$$\begin{aligned} \text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta} - N_t \geq 1) &= 1 - \exp \left[-\mu\Delta - (\lambda_t - \mu) \frac{1 - e^{-\beta\Delta}}{\beta} \right] \\ &= \lambda_t \Delta + O(\Delta^2). \end{aligned} \quad (\text{D16})$$

Before the first new arrival, the intensity is bounded by $\max\{\lambda_t, \mu\}$; between the first and second new arrivals, it is bounded by $\max\{\lambda_t, \mu\} + A_L$. Integrating the two successive conditional arrival densities gives

$$\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta} - N_t \geq 2) \leq \frac{1}{2} \max\{\lambda_t, \mu\} (\max\{\lambda_t, \mu\} + A_L) \Delta^2. \quad (\text{D17})$$

Subtracting this probability from the first-arrival probability yields

$$\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta} - N_t = 1) = \lambda_t \Delta + O(\Delta^2), \quad \text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta} - N_t \geq 2) = O(\Delta^2). \quad (\text{D18})$$

On $\{N_{t+\Delta} = N_t\}$, (D3) and the bounded derivative of $x \mapsto \sqrt{w_0 + w_\lambda x}$ on $[0, \infty)$ imply

$$\lambda_{t+\Delta} = \lambda_t + O(\Delta), \quad Y_{t+\Delta} = Y_t + O(\Delta). \quad (\text{D19})$$

On $\{N_{t+\Delta} - N_t = 1\}$, let $\tau \in (t, t + \Delta]$ be the unique new arrival time. The pre-arrival state follows (D3), so, uniformly in τ ,

$$\lambda_{\tau-} = \mu + (\lambda_t - \mu)e^{-\beta(\tau-t)} = \lambda_t + O(\Delta), \quad (\text{D20})$$

The arrival adds A_L to the state. Applying the deterministic flow after that arrival gives the exact one-arrival terminal state

$$\lambda_{t+\Delta} = \mu + (\lambda_t - \mu)e^{-\beta\Delta} + A_L e^{-\beta(t+\Delta-\tau)}. \quad (\text{D21})$$

Consequently, uniformly in the arrival time τ ,

$$\lambda_{t+\Delta} = \lambda_t + A_L + O(\Delta), \quad Y_{t+\Delta} = Y_t^{(1)} + O(\Delta). \quad (\text{D22})$$

For an OTM put, fix $0 < K < Y_t$ and define the algebraic strike threshold

$$\lambda_K := \frac{K^2 - w_0}{w_\lambda}. \quad (\text{D23})$$

This threshold need not be an attainable nonnegative intensity; when $\lambda_K < 0$, nonnegativity alone rules out a positive put payoff. Since $Y_t^2 = w_0 + w_\lambda \lambda_t$ and $w_\lambda > 0$, the fixed-OTM condition $K < Y_t$ implies $\lambda_K < \lambda_t$. The explicit solution of the fixed-increment linear Hawkes intensity over $[t, t + \Delta]$ is

$$\lambda_{t+\Delta} = \mu + (\lambda_t - \mu)e^{-\beta\Delta} + A_L \int_{(t, t+\Delta]} e^{-\beta(t+\Delta-u)} dN_u. \quad (\text{D24})$$

Because $A_L > 0$, the jump term is nonnegative on every path. Hence

$$\lambda_{t+\Delta} \geq \mu + (\lambda_t - \mu)e^{-\beta\Delta} = \lambda_t + O(\Delta). \quad (\text{D25})$$

The deterministic lower bound in (D25) is continuous in Δ and equals $\lambda_t > \lambda_K$ at $\Delta = 0$. Hence, for all sufficiently small positive Δ , independently of the number and timing of new arrivals,

$$Y_{t+\Delta} = \sqrt{w_0 + w_\lambda \lambda_{t+\Delta}} > K, \quad (\text{D26})$$

almost surely. The nonnegative sum of arrival-induced intensity increments in (D24) keeps the terminal state above the deterministic no-arrival lower bound. Once that lower bound exceeds the strike threshold, every arrival history has zero put payoff. Thus, for all sufficiently small positive Δ ,

$$\text{Put}_t(K, \Delta) = e^{-r\Delta} \mathbb{E}_t^{\mathbb{Q}}[(K - Y_{t+\Delta})^+] = 0 = o(\Delta), \quad K < \text{VIX}_t. \quad (\text{D27})$$

This proves (22); in fact, the fixed-OTM put price is exactly zero on a sufficiently short interval of maturities. The strict gap $Y_t - K > 0$ supplies that interval.

For an OTM call, fix $K > Y_t$. The no-jump payoff is zero for all sufficiently small Δ . On the one-jump event, Lipschitz continuity of $(y - K)^+$ and (D22) give

$$(Y_{t+\Delta} - K)^+ = (Y_t^{(1)} - K)^+ + O(\Delta), \quad \text{on } \{N_{t+\Delta} - N_t = 1\}. \quad (\text{D28})$$

The required payoff moment follows directly from the intensity second moment. Applying the compensated jump formula to λ^2 , first with bounded-state stopping and then using the conditional moment bounds above to remove the stopping, gives

$$\begin{aligned} \frac{d}{ds} \mathbb{E}_t^{\mathbb{Q}}[\lambda_{t+s}^2] &= (2\beta\mu + A_L^2)H_t(s) \\ &\quad - 2(\beta - A_L)\mathbb{E}_t^{\mathbb{Q}}[\lambda_{t+s}^2], \quad \mathbb{E}_t^{\mathbb{Q}}[\lambda_t^2] = \lambda_t^2. \end{aligned} \quad (\text{D29})$$

Together with (D8), this yields locally bounded conditional second moments of λ .

Since $Y_{t+s}^4 = (w_0 + w_\lambda \lambda_{t+s})^2$, $\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+s} + K)^4]$ is also locally bounded. Conditional Hölder's inequality with exponents 4 and 4/3 and (D18) therefore give

$$\begin{aligned} &\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta} + K)\mathbf{1}_{\{N_{t+\Delta} - N_t \geq 2\}}] \\ &\leq \left(\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta} + K)^4]\right)^{1/4} \left(\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta} - N_t \geq 2)\right)^{3/4} \\ &= O(\Delta^{3/2}) = o(\Delta). \end{aligned} \quad (\text{D30})$$

The uniform $O(\Delta)$ payoff error in (D28) contributes $O(\Delta^2)$ after multiplication by the one-arrival probability in (D18). Adding the zero no-arrival payoff and the $o(\Delta)$ contribution in (D30) gives

$$\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta} - K)^+] = \lambda_t \Delta (Y_t^{(1)} - K)^+ + o(\Delta). \quad (\text{D31})$$

Multiplying by $e^{-r\Delta}$ yields

$$\text{Call}_t(K, \Delta) = e^{-r\Delta} \lambda_t \Delta \left(\sqrt{w_0 + w_\lambda(\lambda_t + A_L)} - K \right)^+ + o(\Delta), \quad K > \text{VIX}_t. \quad (\text{D32})$$

This proves (23).

The order- Δ call coefficient is $\lambda_t(Y_t^{(1)} - K)^+$. It is strictly positive exactly when $\lambda_t > 0$ and $Y_t < K < Y_t^{(1)}$; in that case the call price is asymptotically equivalent to its displayed leading term. If $\lambda_t = 0$ or $K \geq Y_t^{(1)}$, the coefficient vanishes and the fixed-OTM call price is $o(\Delta)$. At the boundary $K = Y_t^{(1)}$, the Lipschitz estimate (D22) remains valid, so the one-arrival contribution is $O(\Delta^2)$ and the conclusion is unchanged. The level $Y_t^{(1)} = K_A^*(t)$ is the limiting one-arrival reachability threshold, not an upper bound on VIX following multiple arrivals. ■

E. Small-maturity asymptotic characterization of the VIX option model in

Theorem 3

Write $Y_u := \text{VIX}_u = \sqrt{X_u}$, where $X_u = \mathbf{a} + \mathbf{c} \times \sigma_u^2 + \mathbf{d} \times \lambda_u^V$. The continuous component satisfies the fixed-gap small-time condition stated before Theorem 3. At a jump time of N^V , only the variance component jumps, and

$$X_u - X_{u-} = \mathbf{c} \xi_V > 0, \quad Y_u = \sqrt{X_{u-} + \mathbf{c} \xi_V} > \sqrt{X_{u-}} = Y_{u-}. \quad (\text{E1})$$

Thus every VIX jump is upward. The variance-jump mark density is

$$q(\xi) = \frac{1}{\mu_J} e^{-\xi/\mu_J} \mathbf{1}_{\{\xi > 0\}}, \quad \mu_J > 0. \quad (\text{E2})$$

Let $\mathcal{N}^V(d\ell, d\xi)$ be the marked variance-jump measure. Conditional on the pre-jump filtration, its compensator is $\lambda_{\ell-}^V q(\xi) d\xi d\ell$.

We first establish a common one-jump expansion. Under the local moment properties of the square-root intensity process in (26) and the exponential moment in (25),

$$\text{Prob}_t^{\mathbb{Q}}(N_{t+\Delta}^V - N_t^V \geq 2) = O(\Delta^2), \quad \mathbb{E}_t^{\mathbb{Q}} \left[\left(1 + \sum_{\tau_i \in (t, t+\Delta]} \xi_{V,i} \right) \mathbf{1}_{\{N_{t+\Delta}^V - N_t^V \geq 2\}} \right] = O(\Delta^2). \quad (\text{E3})$$

Hence multi-jump paths contribute $o(\Delta)$ to either fixed-strike payoff.⁸

Fix $g_K(y) = (y - K)^+$ with $K > Y_t$, or $g_K(y) = (K - y)^+$ with $K < Y_t$. On the no-arrival event, the process agrees with $Y^{\text{continuous}}$ driven by the same Brownian motions. The nonnegative no-arrival payoff contribution is therefore bounded by the corresponding jump-suppressed expectation and is $o(\Delta)$ by the maintained fixed-gap condition.

Let $\tau = \inf\{u > t : N_u^V > N_t^V\}$ be the first variance-jump arrival after t . Compensation and the conditional second-moment bounds used in the preceding footnote give, for $p \in \{1, 2\}$,

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}} \left[(1 + Y_\tau)^p \mathbf{1}_{\{\tau \leq t+\Delta\}} \right] \\ &= \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{t+\Delta} \lambda_{\ell-}^V \mathbf{1}_{\{N_{\ell-}^V = N_t^V\}} \int_0^\infty \left(1 + \sqrt{X_{\ell-} + c\xi} \right)^p q(\xi) d\xi d\ell \right] = O(\Delta). \end{aligned} \quad (\text{E4})$$

Indeed, for these values of p the mark integral is bounded by $C^\dagger(1 + X_{\ell-} + \mu_J)$, and $\mathbb{E}_t^{\mathbb{Q}}[\lambda_{\ell-}^V X_{\ell-}]$ is locally bounded by Cauchy–Schwarz. Here and below, $C^\dagger$ denotes a finite positive constant that may change from line to line and may depend on the fixed parameters, the time- t conditioning state, and a fixed upper time horizon, but not on Δ , the arrival time, or the realized jump mark.

Restart the diffusion from its state immediately after τ , using the same Brownian motions and sup-

⁸For $u \geq t$, write $M_u = N_u^V - N_t^V$ and $R_u = \sum_{\tau_i \in (t, u]} \xi_{V,i}$, and set $h(u) = \mathbb{E}_t^{\mathbb{Q}}[M_u \lambda_u^V]$. Since λ^V is continuous, the product rule applied to (26), justified first by localization, and the marked compensator give

$$h'(u) = \kappa_\lambda \theta_\lambda \mathbb{E}_t^{\mathbb{Q}}[M_u] - \kappa_\lambda h(u) + \mathbb{E}_t^{\mathbb{Q}}[(\lambda_u^V)^2], \quad h(t) = 0.$$

The square-root conditional moments are locally bounded, so $\mathbb{E}_t^{\mathbb{Q}}[M_u] = O(u - t)$ and $h(u) = O(u - t)$. The same calculation, using the conditional mark mean μ_J , gives $\mathbb{E}_t^{\mathbb{Q}}[R_u \lambda_u^V] = \mu_J h(u)$. Consequently, compensation yields

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}[M_{t+\Delta}(M_{t+\Delta} - 1)] &= 2 \int_t^{t+\Delta} h(u) du = O(\Delta^2), \\ \mathbb{E}_t^{\mathbb{Q}}[R_{t+\Delta}(M_{t+\Delta} - 1)] &= 2\mu_J \int_t^{t+\Delta} h(u) du = O(\Delta^2). \end{aligned}$$

The asserted probability and mark-weighted bounds follow from

$$(1 + R_u) \mathbf{1}_{\{M_u \geq 2\}} \leq \frac{1}{2} M_u (M_u - 1) + R_u (M_u - 1).$$

Finally, $Y_u^4 = X_u^2$ and the conditional second-moment bounds for $(\sigma_u^2, \lambda_u^V)$ under (24)–(26) give a locally bounded fourth conditional moment of $Y_u + K$. Hölder's inequality therefore bounds either multi-jump payoff contribution by $O(\Delta^{3/2}) = o(\Delta)$.

pressing subsequent variance jumps. Since $X \geq a > 0$, Itô's formula for $Y = \sqrt{X}$ gives bounded continuous diffusion coefficients and a drift bounded in absolute value by $C^\dagger(1 + Y)$. Its conditional expected displacement over a remaining interval of length at most Δ is consequently bounded by $C^\dagger(1 + Y_\tau)\sqrt{\Delta}$. On the exactly-one-arrival event this restarted process agrees with Y . Since either payoff is one-Lipschitz,

$$\begin{aligned} \mathbb{E}_t^\mathbb{Q} \left[|g_K(Y_{t+\Delta}) - g_K(Y_\tau)| \mathbf{1}_{\{N_{t+\Delta}^V - N_t^V = 1\}} \right] \\ \leq C^\dagger \sqrt{\Delta} \mathbb{E}_t^\mathbb{Q} [(1 + Y_\tau) \mathbf{1}_{\{\tau \leq t+\Delta\}}] = O(\Delta^{3/2}). \end{aligned} \quad (\text{E5})$$

The displacement estimate is applied conditional on $\mathcal{F}_\tau$ to the restarted diffusion without conditioning on subsequent arrivals; the exactly-one-arrival indicator is bounded by $\mathbf{1}_{\{\tau \leq t+\Delta\}}$. The immediate first-arrival payoff on paths with at least two arrivals is also $O(\Delta^{3/2})$, by Cauchy–Schwarz, the $p = 2$ estimate above, and the two-arrival probability bound in (E3). Together with the terminal multi-arrival payoff bound established after (E3), these estimates justify the first-arrival reduction over the full mark support. Applying the marked compensator therefore gives

$$\mathbb{E}_t^\mathbb{Q} [g_K(Y_{t+\Delta})] = \mathbb{E}_t^\mathbb{Q} \left[\int_t^{t+\Delta} \lambda_{\ell-}^V \mathbf{1}_{\{N_{\ell-}^V = N_t^V\}} \int_0^\infty g_K(\sqrt{X_{\ell-}} + c\xi) q(\xi) d\xi d\ell \right] + o(\Delta) \quad (\text{E6})$$

$$= \lambda_t^V \Delta \int_0^\infty g_K(\sqrt{X_t} + c\xi) q(\xi) d\xi + o(\Delta). \quad (\text{E7})$$

The indicator selects the first variance-jump arrival. For either payoff, the map

$$x \mapsto \int_0^\infty g_K(\sqrt{x + c\xi}) q(\xi) d\xi \quad (\text{E8})$$

is Lipschitz on $[a, \infty)$, with constant at most $1/(2\sqrt{a})$. Conditional L^2 continuity of X and λ^V at t , their locally bounded conditional second moments, and $\text{Prob}_t^\mathbb{Q}(N_{\ell-}^V > N_t^V) = O(\ell - t)$ give conditional L^1 convergence of the full compensator integrand at t . Averaging this convergence over $(t, t + \Delta]$ proves the second equality in (E7).

For a fixed OTM put, $K < Y_t = \sqrt{X_t}$. Since $\sqrt{X_t} + c\xi > K$ for every $\xi > 0$, the put integrand in (E7) is identically zero. Thus

$$\text{Put}_t(K, \Delta) = e^{-r\Delta} \mathbb{E}_t^\mathbb{Q} [(K - Y_{t+\Delta})^+] = o(\Delta), \quad K < Y_t. \quad (\text{E9})$$

For a fixed OTM call, $K > Y_t$, (E7) gives

$$\mathbb{E}_t^{\mathbb{Q}}[(Y_{t+\Delta} - K)^+] = \lambda_t^V \Delta I(K, X_t) + o(\Delta), \quad (\text{E10})$$

where

$$I(K, x) = \int_0^\infty \left(\sqrt{x + c\xi} - K \right)^+ \frac{1}{\mu_J} e^{-\xi/\mu_J} d\xi. \quad (\text{E11})$$

Set $\varphi := c\mu_J$. Because $K > \sqrt{x}$, the payoff is positive exactly when

$$\xi > \frac{K^2 - x}{c}. \quad (\text{E12})$$

Consequently,

$$I(K, x) = \int_{(K^2 - x)/c}^\infty \left(\sqrt{x + c\xi} - K \right) \frac{1}{\mu_J} e^{-\xi/\mu_J} d\xi. \quad (\text{E13})$$

With $u = \sqrt{x + c\xi}$,

$$I(K, x) = \frac{2}{\varphi} e^{x/\varphi} \int_K^\infty u(u - K) e^{-u^2/\varphi} du.$$

Let $a_K = K/\sqrt{\varphi}$. The required Gaussian-tail integrals are

$$\int_K^\infty u^2 e^{-u^2/\varphi} du = \varphi^{3/2} \left\{ \frac{a_K}{2} e^{-a_K^2} + \frac{\sqrt{\pi}}{4} \operatorname{erfc}(a_K) \right\}, \quad (\text{E14})$$

and

$$\int_K^\infty u e^{-u^2/\varphi} du = \frac{\varphi}{2} e^{-a_K^2}. \quad (\text{E15})$$

Since $K = \sqrt{\varphi} a_K$, the elementary exponential terms cancel:

$$\int_K^\infty u(u - K) e^{-u^2/\varphi} du = \varphi^{3/2} \frac{\sqrt{\pi}}{4} \operatorname{erfc}(a_K). \quad (\text{E16})$$

Therefore,

$$I(K, x) = e^{x/\varphi} \frac{\sqrt{\pi\varphi}}{2} \operatorname{erfc}\left(\frac{K}{\sqrt{\varphi}}\right) = \exp\left(\frac{x}{c\mu_J}\right) \frac{\sqrt{\pi c\mu_J}}{2} \operatorname{erfc}\left(\frac{K}{\sqrt{c\mu_J}}\right). \quad (\text{E17})$$

Multiplying by $e^{-r\Delta}$ and setting $x = X_t = Y_t^2 = \text{VIX}_t^2$ gives

$$\text{Call}_t(K, \Delta) = e^{-r\Delta} \lambda_t^V \Delta \exp\left(\frac{\text{VIX}_t^2}{c\mu_J}\right) \frac{\sqrt{\pi c\mu_J}}{2} \operatorname{erfc}\left(\frac{K}{\sqrt{c\mu_J}}\right) + o(\Delta), \quad K > \text{VIX}_t. \quad (\text{E18})$$

If $\lambda_t^V > 0$, the displayed coefficient is strictly positive because the exponential jump law assigns positive

probability to every interval above the crossing threshold. This proves Theorem 3. ■

F. Realized tails and signed jump diagnostics

We complement the option-price diagnostic in Remark 1 with evidence on realized log changes in nearest-maturity VIX futures (under the data-generating measure). Let h_t^{vix} denote a 30-minute or five-minute log return from the Barchart continuous-contract series for the nearest-maturity VIX futures.

Remark 2 (Peaks-over-threshold implication) *Let X have distribution function F , survival function $\bar{F}(x) := \text{Prob}(X > x)$, and right endpoint $x_F := \sup\{x : F(x) < 1\}$. Suppose there is a positive scaling function $\omega(u)$ such that, as $u \uparrow x_F$, the conditional distribution of $(X - u)/\omega(u)$ given $X > u$ converges to a nondegenerate generalized Pareto law with shape parameter ζ and unit scale. If $\zeta > 0$, then $x_F = \infty$ and $\bar{F}$ is regularly varying with index $-1/\zeta$: $\bar{F}(x) = x^{-1/\zeta} \mathbb{L}(x)$ as $x \rightarrow \infty$, for a slowly varying function $\mathbb{L}$; see Embrechts, Klüppelberg, and Mikosch (1997, Chapters 3 and 4).*

Peaks-over-threshold construction. The peaks-over-threshold estimates characterize return-tail behavior. For $(\tau^-, \tau^+) \in \{(0.10, 0.90), (0.05, 0.95)\}$, let u^- and u^+ be the corresponding empirical quantiles of $\{h_t^{\text{vix}}\}$. Define lower-tail exceedances $Z_i^- = u^- - h_{t_i}^{\text{vix}}$ when $h_{t_i}^{\text{vix}} < u^-$ and upper-tail exceedances $Z_i^+ = h_{t_i}^{\text{vix}} - u^+$ when $h_{t_i}^{\text{vix}} > u^+$. We estimate $(\hat{\zeta}^-, \hat{\omega}^-)$ and $(\hat{\zeta}^+, \hat{\omega}^+)$ by maximum likelihood on the two exceedance samples. Because the intraday returns are dependent, Appendix Table A1 reports 90% stationary-block-bootstrap confidence intervals based on 500 replications.⁹

All four downside-shape estimates in Appendix Table A1 are positive, and their reported 90% bootstrap intervals exclude zero. For example, for 30-minute returns and $\tau^- = 0.10$, $\hat{\zeta}^- = 0.303$ with interval $[0.273, 0.336]$. Under the peaks-over-threshold approximation in Remark 2, the estimates describe a heavy left tail in nearest-maturity VIX-futures returns. The upper-tail estimates are larger in every specification:

⁹The 30-minute and five-minute samples contain 58,890 and 325,340 nonmissing nearest-maturity VIX futures log returns, respectively. For $(\tau^-, \tau^+) = (0.10, 0.90)$, the 30-minute sample contains 5,891 lower-tail and 5,889 upper-tail exceedances, while the five-minute sample contains 32,493 lower-tail and 32,526 upper-tail exceedances. For $(\tau^-, \tau^+) = (0.05, 0.95)$, the 30-minute sample contains 2,942 lower-tail and 2,946 upper-tail exceedances, while the five-minute sample contains 16,263 lower-tail and 16,271 upper-tail exceedances. Exceedances use strict threshold inequalities. The reported 90% confidence intervals use 500 retained replications of a stationary-block bootstrap applied to the signed return series, with expected block length 20 return observations. Both empirical thresholds and both GPD fits are re-estimated in every replication. Shape and scale intervals are percentile intervals; the interval for $\hat{\zeta}^- - \hat{\zeta}^+$ uses bootstrap- t inversion with likelihood-based studentization.

the reported intervals for $\hat{\zeta}^- - \hat{\zeta}^+$ lie strictly below zero. Thus both tails are estimated to be heavy, with the right tail heavier than the left over the reported sampling frequencies and thresholds.¹⁰

Evidence from signed high-frequency jump diagnostics. To examine jump directions at the five-minute frequency, we apply the Lee and Mykland (2008) procedure to nearest-maturity VIX-futures returns.¹¹ At the nominal 5% familywise level, the procedure flags 396 five-minute intervals: 149 negative and 247 positive.¹²

¹⁰The signs of the estimated shape parameters and their ordering are unchanged across the reported thresholds, although the five-minute shape estimates increase when the exceedance fraction is reduced from 10% to 5%.

¹¹Writing h_i for the five-minute return, we set $K_{LM} = 270$ and use $L_i = h_i / \hat{\sigma}_i$, where $\hat{\sigma}_i^2 = (K_{LM} - 2)^{-1} \sum_{j \in \mathcal{B}_i} |h_j| |h_{j-1}|$. The set $\mathcal{B}_i$ indexes the $K_{LM} - 2 = 268$ most recent products strictly before i , each formed from adjacent retained returns on the same calendar date. The window carries across dates, omitting cross-date products; testing requires a complete window and a positive denominator. The bipower average is unannualized, with no $\pi/2$ multiplier or time-of-day rescaling. For n valid tests over the full sample and $c = \sqrt{2/\pi}$, let $C_n = \frac{\sqrt{2 \log n}}{c} - \frac{\log \pi + \log(\log n)}{2c\sqrt{2 \log n}}$ and $S_n = \frac{1}{c\sqrt{2 \log n}}$. We flag interval i when $(|L_i| - C_n)/S_n > \beta_{0.05}$, where $\beta_{0.05} = -\log[-\log(0.95)]$ is the 95th percentile of the standard Gumbel distribution. The nominal asymptotic 5% familywise level applies jointly to all valid intervals and both return signs in this single futures series.

¹²Todorov, Tauchen, and Gryn timer (2014) study volatility-jump activity and asymmetry using high-frequency VIX and S&P 500 futures data; their parametric volatility model permits jumps of either sign. We derive short-maturity VIX option-price asymptotics under the signed, state-dependent stock-jump intensity update in (13).

Appendix Table A1

Peaks-over-threshold estimation results for tails of 30-minute and five-minute VIX futures returns

This table reports peaks-over-threshold estimates for lower and upper tails of nearest-maturity VIX futures log returns, sampled at 30-minute and five-minute intervals. Threshold pairs are $(\tau^-, \tau^+) = (0.10, 0.90)$ and $(0.05, 0.95)$, with exceedances defined as in Appendix F. The reported parameters are the shape ζ and scale ω ; $\zeta > 0$ indicates a heavy tail. Parameters $(\hat{\zeta}^-, \hat{\omega}^-)$ and $(\hat{\zeta}^+, \hat{\omega}^+)$ are estimated by maximum likelihood. For $\zeta \neq 0$, the log-likelihood is $\mathcal{L}(\zeta, \omega) = -n \log \omega - (1 + 1/\zeta) \sum_{i=1}^n \log(1 + \zeta Z_i/\omega)$; at $\zeta = 0$ it is interpreted by the exponential-limit likelihood $\mathcal{L}(0, \omega) = -n \log \omega - \sum_{i=1}^n Z_i/\omega$. We enforce $\omega > 0$ and $1 + \zeta Z_i/\omega > 0$ for every exceedance. Bootstrap entries are 90% stationary-block-bootstrap confidence intervals based on 500 replications. The final column reports the estimated shape difference $\hat{\zeta}^- - \hat{\zeta}^+$ and its corresponding confidence interval. The sample runs from April 20, 2009, through December 31, 2025, with 4203 trading days. The 30-minute (respectively, five-minute) series contains 58,890 (respectively, 325,340) observations. The continuous series follows the vendor's roll convention; the reported statistics describe this constructed VIX futures series.

	quantile level: $\tau^- = 0.10$		quantile level: $\tau^+ = 0.90$		Shape difference
	$\hat{\zeta}^-$	$\hat{\omega}^-$	$\hat{\zeta}^+$	$\hat{\omega}^+$	$\hat{\zeta}^- - \hat{\zeta}^+$
<u>Estimates for 30-minute VIX futures returns</u>					
Estimate	0.303	0.0065	0.396	0.0074	-0.093
90% bootstrap CI	[0.273, 0.336]	[0.0062, 0.0067]	[0.368, 0.427]	[0.0071, 0.0078]	[-0.127, -0.055]
<u>Estimates for five-minute VIX futures returns</u>					
Estimate	0.335	0.00247	0.369	0.00251	-0.034
90% bootstrap CI	[0.320, 0.350]	[0.00243, 0.00251]	[0.353, 0.385]	[0.00247, 0.00256]	[-0.052, -0.018]
	quantile level: $\tau^- = 0.05$		quantile level: $\tau^+ = 0.95$		Shape difference
	$\hat{\zeta}^-$	$\hat{\omega}^-$	$\hat{\zeta}^+$	$\hat{\omega}^+$	$\hat{\zeta}^- - \hat{\zeta}^+$
<u>Estimates for 30-minute VIX futures returns</u>					
Estimate	0.314	0.0079	0.411	0.0097	-0.097
90% bootstrap CI	[0.272, 0.360]	[0.0075, 0.0083]	[0.368, 0.449]	[0.0090, 0.0103]	[-0.154, -0.047]
<u>Estimates for five-minute VIX futures returns</u>					
Estimate	0.412	0.00280	0.481	0.00274	-0.069
90% bootstrap CI	[0.393, 0.432]	[0.00271, 0.00288]	[0.460, 0.500]	[0.00266, 0.00283]	[-0.097, -0.047]