

Deleveraging When Investors Trade Again

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POVZETEK: Članek raziskuje, ali znanje o prihodnjih trgovinah dovoli za izbiro alokacije, ki minimalizira očekovane nedovoljene sredstva računov. V treh-oddevnichiški ekonomiji s dvema hederji in eno preostalo rizikno nosilce, institucija deloži odrejen ukaz o razkršču med hederji. Cene, trgovine in položaji se ujemajo na skupnem regularnem ekvilibrijsnem obsegu, vendar imajo različne minimalne alokacije. Cene sredstev spremenijo terminalne stanja, ki potrebujejo ponovno zaopatitev, brez spremembe neogranicene trgovinske potrebe. Ključne ugotovitve vključujejo, da je informacija o kontrahentovih nedovoljenih sredstvih ključna za izbiro alokacije, če je poznana razdelitev odzivov.

METODA: Članek razvija teorijo o treh-oddevnichiških ekonomiji, kjer hederji in kontrahenti izbirajo svoje položaje pod omejitvami z dolgoročnimi sredstvi, položaji in količine kontrahentov. Cene, trgovine in položaji se ujemajo na skupnem regularnem ekvilibrijsnem obsegu, vendar imajo različne minimalne alokacije. Ključno je, da informacija o kontrahentovih nedovoljenih sredstvih je ključna za izbiro alokacije, če je poznana razdelitev odzivov.

KLJUČNE UGOTOVITVE:

- V treh-oddevnichiški ekonomiji s dvema hederji in eno preostalo rizikno nosilce, skupne cene, trgovine in položaji ujemajo na skupnem regularnem ekvilibrijsnem obsegu, vendar imajo različne minimalne alokacije.
- Cene sredstev spremenijo terminalne stanja, ki potrebujejo ponovno zaopatitev, brez spremembe neogranicene trgovinske potrebe.
- Informacija o kontrahentovih nedovoljenih sredstvih je ključna za izbiro alokacije, če je poznana razdelitev odzivov.

POMEN ZA TRGOVANJE: Ugotovitve poudarjajo, da je informacija o kontrahentovih nedovoljenih sredstvih ključna za izbiro alokacije, ki minimalizira očekovane nedovoljene sredstva računov. To je relevantno za praktiko, saj pomeni, da institucije morajo upoštevati te informacije pri odločanju o alokaciji, če želijo minimizirati finančne težave računov.

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Abstract

A clearing institution that must terminate a fixed quantity of contracts has to decide which accounts bear the reduction. We study whether knowing subsequent trading is enough to choose the allocation minimizing expected account funding shortfalls. In a three-participant economy, we construct two cash-buffer states with identical prices, trades and positions under every allocation on a common regular equilibrium branch. Their shortfall objectives are strictly convex but have different unique minimizers. Counterparty cash changes the states requiring replenishment without changing unconstrained demand. We characterize when this missing cash state leaves a feasible allocation derivative unchanged. Under additional one-factor tail conditions, a unique cash threshold separates opposite local priorities when the counterparty's marginal shortfall contribution crosses the other accounts' benefit. With a known payoff distribution, the counterparty's shortfall cutoff supplies the missing state information. Equal fixed-price retention shares yield a separable, strictly convex benchmark on a fully regular allocation domain. The common-map construction also persists under a class of small nonnormal payoff perturbations. The criterion measures terminal account replenishment under stated settlement conditions. The results distinguish information needed to predict trading from information needed to evaluate funding consequences, within the specified branches and observation set.

Keywords: auto-deleveraging; contract termination; re-trading; cash buffers; funding shortfalls; clearing; information sufficiency.

Declaration of generative AI use

The author used OpenAI Codex for language polishing and routine software engineering support, including writing repetitive code and assisting with computational implementation. The research design, theoretical arguments, and interpretation of results were reviewed by the author, who takes full responsibility for the final manuscript.

1 Introduction

A clearing institution that faces a fixed quantity of contracts to terminate must decide how to divide the reduction among surviving accounts. Those accounts may trade again. An investor can rebuild a hedge, reduce another position, or accept a different exposure; someone must take the other side of each new trade. Evaluating a termination allocation therefore requires more than comparing positions immediately after intervention.

This paper asks: even if the institution knew the complete price and trading response to every feasible allocation, would that information identify the allocation minimizing expected account funding shortfalls? The distinction matters because predicting behavior and evaluating its funding consequences are different tasks. A counterparty can execute the same trade with two different cash buffers, yet need funds in different terminal payoff states. The institution may then prefer different allocations even though the observed response to each allocation is unchanged.

We study a three-participant continuation economy with two hedgers and one surviving risk-bearing counterparty. The institution allocates a prescribed total termination between the hedgers. Old contracts are cash-marked before new trades enter at the same price. Participants choose their new positions optimally under cash margin, position and counterparty quantity limits, and the market clears. At final contractual settlement, the objective sums the expected amount by which each of the three cleared accounts falls below zero. Execution costs are actual cash outflows, and the counterparty's funding shortfall is included.

The main result constructs two economies that differ only in counterparty cash. Their prices, trades and positions coincide over the entire original allocation interval on a common regular branch, meaning that all relevant trading constraints are strictly slack. Nevertheless, each branch objective is strictly convex and has a unique minimizing allocation, and the two minimizers differ. In the original construction they lie on opposite sides of the same interior allocation. Any rule given the stated structure and that complete common map, but not cash or a quantity revealing it, receives identical inputs in the two economies and cannot select both optima. This is an existence result for a specified information set and branch, not a statement that every change in cash changes the preferred allocation.

The mechanism comes from the states in which replenishment is required. When adding cash shifts a participant's certainty equivalent by a constant, it leaves unconstrained trading demand unchanged. As long as the common choices remain feasible, the trading map also stays unchanged. Cash nevertheless moves the boundary between terminal states with and without an account shortfall. We show that a feasible allocation derivative is invariant over a cash-information interval exactly when the counterparty's conditional mean marginal resource change vanishes over the states swept by that boundary, weighted by their density. The result allows the resource change to include all price, position and execution-cost effects.

Derivative invariance and a common preferred direction are different questions. Under one-factor tail conditions, the counterparty's marginal shortfall contribution is positive and decreases with cash. If it opposes the other accounts' marginal benefit and crosses it, a unique cash threshold separates opposite local priorities. If it does not cross, cash can alter the derivative's size without reversing its sign. The threshold concerns a fixed allocation and direction. With a known payoff distribution and account structure, observing the counterparty's shortfall cutoff supplies the missing state information; that cutoff can also recover cash itself.

There is also a positive allocation benchmark. When both hedgers leave the same fixed-price fraction of termination in place after rebuilding, aggregate demand depends on the total task rather than its division. On a comparison domain that remains entirely regular, price and counterparty resources are constant across allocations. The objective then separates into strictly

convex account terms, and its minimizer equalizes marginal benefits subject to bounds. This is one concrete way for the missing counterparty cash to be irrelevant to allocation comparisons. It is not necessary for every form of information sufficiency.

Finally, the common-map construction persists under a class of small, mean-zero, unit-variance nonnormal payoff perturbations. The proof uses the appropriate CARA log-moment-generating function, not the normal demand formula outside its domain. Uniform feasibility, curvature and derivative-sign margins continue the original conclusion. This establishes a nonempty class beyond a single Gaussian example while retaining cash-translation-invariant preferences.

The funding criterion has a conditional operational meaning. When counted cash is available at final settlement, no positive margin floor remains, and accounts cannot freely draw on each other’s surpluses or outside assets by the deadline, the sum of account shortfalls is the cash that must be supplied to meet their contractual obligations. It measures replenishment principal even when all obligations are ultimately paid. Section 2.4 states these conditions and distinguishes this amount from a welfare or credit-loss measure. The model describes a continuation problem after a prescribed termination task.

1.1 Relation to existing work

Campbell et al. (2026, Theorem 2.4 and Proposition 3.4) study risk-based ADL with a fixed termination quantity. Their static single-asset analysis yields a transparent leverage-equalization rule; their multi-asset expected-loss problem uses asset shadow prices to separate accounts. The static setup holds allocations fixed to the evaluation horizon without subsequent participant trading. Freezing new trades at zero in our continuation gives a separable short-account objective with a constant long-side term. With optimal re-trading, both positions and cash intercepts respond to allocation. Our equal-retention benchmark specifies when separation survives that response. It does not inherit the static rule’s stronger statewise, distribution-free or path-independence properties.

Bastide et al. (2024) analyze portfolio choice and price formation around clearing-member default. Their Section 3.2 distinguishes settlement payments on old contracts and new hedges, and Section 6 combines market equilibrium with credit, funding and capital evaluation. The private Gaussian trading problem here can be written in their entropic Radner form even with positive quadratic execution costs; Appendix F gives the exact change of variables. This correspondence is between private trading problems at a given allocation. It does not remove real costs from our shortfall objective or make the two policy problems identical. Their Appendix D identifies incorporating credit costs into private equilibrium as a further step. Our funding criterion retains the private payoffs specified below.

Relatedly, Bastide et al. (2023, Section 7) evaluate the porting of a defaulted member’s client portfolio by summing incremental valuation adjustments across surviving members, including the potential taker. Thus considering the receiving participant’s risk and distinguishing trading quantities from subsequent risk evaluation are already part of clearing research. Their choice of a portfolio taker and all-in valuation criterion differ from the fixed-quantity termination allocation and terminal account-funding criterion studied here.

The use of a positive-part cash shortfall also has a direct precedent. Bardoscia et al. (2025, Section 2.4, equation (7)) measure centrally cleared payment obligations exceeding initially available cash before further funding actions. This supports the interpretation of a funding quantity, while their payment process and allocation objects differ from ours. On price effects, Dávila and Korinek (2018, Proposition 2) distinguish distributive and collateral externalities and separate marginal valuations, trading positions and equilibrium price responses. Our local accounting likewise evaluates responses with marginal shortfall weights. Those weights are not

the welfare valuations in their analysis, and the resulting formula does not imply an optimal tax or welfare theorem.

The paper combines a full-domain allocation construction with a description of the shortfall states that make cash relevant for allocation. On a common trading map, it identifies when cash leaves a local comparison unchanged and when it reverses the preferred direction. The argument uses cash translation and standard differentiation of positive-part losses; the results concern the specified funding objective and observation set.

Section 2 defines the economy, criterion and observations. Section 3 gives the main construction. Section 4 develops the cash mechanism, local threshold and cutoff information. Section 5 provides the separable benchmark, and Section 6 states nonnormal persistence and economic scope. Proofs, notation and reproduction details are in the appendices.

2 Environment and Funding Criterion

This section defines the normal baseline. Section 4 states the distributional and common-map conditions needed for its mechanism results, and Section 6 specifies the nonnormal continuation.

2.1 Participants and the fixed termination task

There are two surviving hedgers, indexed by $i \in \{1, 2\}$, and one surviving counterparty, indexed by D . The counterparty is a participant with its own risk exposure, trading cost and funding constraints. It supplies the long side of the surviving and newly traded contracts; it is not the clearing institution or an unlimited outside source of risk-bearing capacity.

Before the prescribed termination, hedger i has q_i short contracts and an outside exposure $H_i X$. Here $X = P_2 - P_0 \sim N(0, \sigma^2)$ is a payoff change, with $\sigma > 0$. It is not a literal nonnegative asset price. The counterparty's outside exposure is $-H_D X$. These outside exposures enter private preferences but are unavailable to fund the cleared accounts at the settlement deadlines in this model.

The institution chooses $d = (d_1, d_2)$ in

$$\mathcal{D} = \{d \in \mathbb{R}^2 : 0 \leq d_i \leq q_i, d_1 + d_2 = Q\}, \quad 0 \leq Q \leq q_1 + q_2. \quad (1)$$

The quantity Q is given. Define

$$a_i = q_i - d_i, \quad A = a_1 + a_2 = q_1 + q_2 - Q. \quad (2)$$

The surviving counterparty holds A old long contracts. The already determined side of the default disposal and its costs lie outside this continuation economy and are held fixed across d . We do not choose the size of the initial intervention or explain the initial default.

At date 0, termination takes place at the common reference value P_0 , at which the contracts have zero current value. Hedger cash E_i and counterparty cash b include both posted and free cash collateral. A reduction in required margin releases previously committed cash for use; it does not add to account wealth. In particular, terminating d_i contracts releases $m_i d_i$ of required margin, where m_i is cash margin per contract.

2.2 Cash settlement, new trades and constraints

At date 1 the price is $P_1 = P_0 - \lambda$. Old positions are first cash-marked to this price: hedger i receives $a_i \lambda$ and the counterparty pays $A \lambda$. These flows add to zero. The same price is used for

the ensuing batch of new trades. There is no bridge credit, new outside funding or price change between these steps.

Hedger i chooses an additional short position r_i . The counterparty chooses an additional long position R . Either choice may be negative, representing further reduction of an old position. Final positions and market clearing satisfy

$$z_i = a_i + r_i \geq 0, \quad z_D = A + R \geq 0, \quad R = r_1 + r_2. \quad (3)$$

The counterparty has a cap $R \leq K$ on its *net additional* long position. This quantity cap is distinct from its cash buffer b .

New contracts have no purchase payment at entry. Trading costs $k_i r_i^2/2$ and $k_D R^2/2$, with $k_j > 0$, are paid to an outside service provider. After cash marking and costs, the model's counted cash amounts are

$$M_i = E_i + a_i \lambda - \frac{k_i r_i^2}{2} \geq m_i z_i, \quad i = 1, 2, \quad (4)$$

$$M_D = b - A \lambda - \frac{k_D R^2}{2} \geq m_D z_D, \quad R \leq K. \quad (5)$$

Margin coefficients are positive. Equations (4)–(5) are the post-trade funding requirements for this batch. They specify the timing convention being analyzed; no additional within-batch margin-restoration rule is imposed.

Given (d, λ) , the hedger's feasible set is

$$\mathcal{F}_i(d, \lambda) = \left\{ r \geq -a_i : E_i + a_i \lambda - \frac{k_i r^2}{2} \geq m_i(a_i + r) \right\}, \quad (6)$$

and the counterparty's feasible set is

$$\mathcal{F}_D(\lambda; b) = \left\{ R \in [-A, K] : b - A \lambda - \frac{k_D R^2}{2} \geq m_D(A + R) \right\}. \quad (7)$$

These sets are convex when nonempty. The quadratic costs also bound feasible trades. The main theorem proves nonemptiness and strict feasibility for its entire allocation domain, rather than assuming them.

2.3 Terminal resources and private choice

At date 2 the contracts settle at payoff change $X + \lambda$ relative to their date-1 reference. Account resources before outside replenishment are

$$e_i = M_i - z_i(X + \lambda) = c_i - z_i X, \quad c_i = E_i - \lambda r_i - \frac{k_i r_i^2}{2}, \quad (8)$$

$$e_D = M_D + z_D(X + \lambda) = c_D + z_D X, \quad c_D = b + \lambda R - \frac{k_D R^2}{2}. \quad (9)$$

The c_j are terminal resource intercepts, not the cash M_j available for date-1 margin. Posted margin is counted once and is available for final settlement. Market clearing implies the statewise identity

$$e_1 + e_2 + e_D = E_1 + E_2 + b - \frac{k_1 r_1^2 + k_2 r_2^2 + k_D R^2}{2}. \quad (10)$$

Thus settlement redistributes cash; the execution costs are actual outflows from the three-account economy.

Participants have full-recourse constant absolute risk aversion (CARA) preferences. Their economic payoffs are $e_i + H_i X$ and $e_D - H_D X$, without truncation at a zero cleared-account balance. Let $\alpha_j = \gamma_j \sigma^2$ for risk-aversion coefficients $\gamma_j > 0$. Normality gives the certainty equivalents

$$U_i(r_i; d, \lambda) = E_i - \lambda r_i - \frac{k_i r_i^2}{2} - \frac{\alpha_i}{2} (H_i - a_i - r_i)^2, \quad (11)$$

$$U_D(R; \lambda, b) = b + \lambda R - \frac{k_D R^2}{2} - \frac{\alpha_D}{2} (A + R - H_D)^2. \quad (12)$$

Each participant treats λ as given and maximizes its certainty equivalent over its own feasible set. An additional emergency financing fee is not included in these preferences. Full recourse preserves the distinction between an account's temporary funding shortfall and the participant's ultimate economic liability.

Definition 1 (Equilibrium and the regular branch). For given (d, b) , an equilibrium is a price adjustment λ and trades (r_1, r_2, R) that maximize (11)–(12) over (6)–(7) and satisfy $r_1 + r_2 = R$. A *regular equilibrium* has strict cash-margin inequalities, $z_1, z_2, z_D > 0$, and $R < K$. A regular branch is a smooth family of such equilibria indexed by the allocation.

Positive quadratic coefficients make each private objective strictly concave. They do not guarantee that the market has only one equilibrium: price also changes the funding sets. We evaluate allocations along an explicitly specified regular branch. We do not assign this branch's value to another equilibrium or rank equilibrium-selection rules. An endpoint of the allocation interval can still support a strictly interior trading equilibrium.

2.4 The account-funding criterion

For a specified branch, write $e_j(d; b, X)$ for the resulting terminal account resources. The institution evaluates

$$J(d; b) = \sum_{j \in \{1, 2, D\}} \mathbb{E}[(-e_j(d; b, X))^+], \quad y^+ = \max\{y, 0\}. \quad (13)$$

The criterion counts all three accounts separately. A surplus in one account is not automatically a free source of immediate funding for another account.

To state its accounting meaning, let $o_i = z_i(X + \lambda)$ and $o_D = -z_D(X + \lambda)$ denote signed net payment obligations. Then $e_j = M_j - o_j$ and, state by state,

$$\sum_j (-e_j)^+ = \min_{\ell_j \geq 0} \left\{ \sum_j \ell_j : M_j + \ell_j \geq o_j \text{ for every } j \right\}. \quad (14)$$

Indeed, each constraint is met at minimum by $\ell_j = (o_j - M_j)^+$. This is an accounting identity for separately funded accounts, not a new financing-choice model. It measures terminal top-ups when all counted cash can be used by the final payment deadline, no positive margin floor remains after the contracts end, contractual receipts arrive as modeled, and outside assets or other accounts' surpluses are not freely available at that deadline.

A related operational measure records centrally cleared payment obligations exceeding available cash; see Bardoscia et al. (2025, Section 2.4, equation (7)). That precedent does not identify (13) with every institution's actual funding requirement. If a continuing position instead had to retain a positive margin balance h_j , the relevant restoration amount would be $(h_j - e_j)^+$, generally different from $(-e_j)^+$. We do not change the criterion to that amount. Nor do

we interpret funding principal as interest cost, permanent unrecoverable loss, welfare, or the clearing house's own liquidity draw.

For normal X , symmetry gives the same shortfall function for long and short accounts:

$$F_\sigma(c, z) = \sigma z \phi\left(\frac{c}{\sigma z}\right) - c \Phi\left(-\frac{c}{\sigma z}\right), \quad z > 0, \quad F_\sigma(c, 0) = (-c)^+. \quad (15)$$

Here ϕ and Φ are the standard normal density and distribution function. Thus $J = F_\sigma(c_1, z_1) + F_\sigma(c_2, z_2) + F_\sigma(c_D, z_D)$ on the chosen branch. Appendix A.3 derives the derivatives used below.

2.5 Observations and the allocation to be identified

Let s collect $q_i, H_i, E_i, \gamma_i, k_i, m_i$, the counterparty's $H_D, \gamma_D, k_D, m_D, K$, the fixed Q , the payoff distribution and the settlement convention. Only counterparty cash b is omitted. Write $\theta_b = (s, b)$. For a specified branch define the labeled map

$$x_{\text{reg}}(d; \theta_b) = (\lambda(d), r_1(d), r_2(d), R(d), z_1(d), z_2(d), z_D(d)). \quad (16)$$

The allowed observation is

$$\mathcal{O}(\theta_b) = (s, \{(d, x_{\text{reg}}(d; \theta_b)) : d \in \mathcal{D}\}). \quad (17)$$

It includes the entire allocation-indexed map, hence all derivatives of that map wherever they exist. It is stronger than observing one realized price and trade. The two hedgers' terminal intercepts can also be recovered from these inputs.

The observation does not include b, M_D, c_D , the numerical cash-margin slack, total system cash, or an equivalent shortfall statistic. For example, $M_D - m_D z_D$ and the map together would reveal b . The truth of strict feasibility can be common even though the numerical slack is different. Excluding these cash-revealing quantities defines the information question; it is not a claim that a real clearing institution cannot obtain them. Participants still use their own budgets in their private choices.

Definition 2 (An observation-based allocation rule). A deterministic rule ψ assigns an allocation in $\mathcal{D}$ to the observation (17). It is optimal on a specified family of economies if

$$\psi(\mathcal{O}(\theta_b)) \in \arg \min_{d \in \mathcal{D}} J(d; b) \quad \text{for every economy in that family,} \quad (18)$$

where each objective uses its designated regular branch.

The family in the main theorem consists of two explicitly specified cash states. Knowing these two possible states does not tell the rule which one is realized. There is no restriction on its computational ability.

Finally, three identification questions must be kept separate. At d_0 in the relative interior of $\mathcal{D}$, two-sided feasible directions belong to $T = \text{span}(\mathcal{D} - d_0)$. For a nondegenerate two-account allocation interval, this is $\{\delta : \delta_1 + \delta_2 = 0\}$; if the domain is a singleton, only the zero direction is feasible. At an endpoint of a nondegenerate interval, a feasible move must point into the interval. The value $D_\delta J$, the sign needed for strict local improvement, and an allocation satisfying (18) are different objects. Unequal derivatives alone do not imply opposite improvement directions. The theorem establishes distinct global minimizers on the stated branch using both strict convexity and opposite signs at a common point.

3 Identical Trading Maps, Different Allocations

Consider the following two economies within Section 2. They use the same parameters except for the counterparty's cash buffer. The parameter values are a construction in model units, not an empirical calibration.

Table 1: The fixed construction

Object	Symbol	Value
Initial shorts and outside exposures	q_1, q_2, H_1, H_2	1, 1, 1, 1
Termination and payoff scale	Q, σ	1, 1
Hedger cash	(E_1, E_2)	(1/2, 3)
Hedger risk aversion	(γ_1, γ_2)	(3, 1)
Hedger cost coefficients	(k_1, k_2)	(1, 3)
Counterparty risk aversion and cost	γ_D, k_D	1/100, 1/100
Counterparty outside exposure and cap	H_D, K	1, 1
Cash margin per contract	m_1, m_2, m_D	1/10, 1/10, 1/10
Counterparty cash (the only difference)	b	1/5 or 4

Write $d = (u, 1 - u)$, with $u \in [0, 1]$. Then $a_1 = 1 - u$, $a_2 = u$ and $A = 1$. Write $J_b(u)$ for (13) evaluated along the regular branch specified in the theorem. The fixed-price retention share is $\tau_i = k_i/(\alpha_i + k_i)$: when termination rises by one unit, this is the portion left in place after optimal rebuilding at a fixed price. Here $\tau_1 = 1/4$ and $\tau_2 = 3/4$. The convexity result below is proved for this heterogeneous-retention construction; it does not follow from an equal-retention benchmark.

Theorem 1 (Common trading map and distinct optimal allocations). *Fix Table 1, with $b \geq 1/5$ when only a lower bound is specified.*

1. For every $u \in [0, 1]$, the following map is a regular equilibrium, independent of b :

$$\lambda(u) = \frac{1 + 2u}{202}, \quad r_1(u) = \frac{604u - 1}{808}, \quad r_2(u) = \frac{201 - 204u}{808}, \quad (19)$$

$$R(u) = \frac{25 + 50u}{101}, \quad z_1(u) = \frac{807 - 204u}{808}, \quad z_2(u) = \frac{201 + 604u}{808}, \quad (20)$$

$$z_D(u) = \frac{126 + 50u}{101}.$$

All trading constraints hold strictly over the entire interval. This is the unique equilibrium with all these constraints slack; equilibria on other constraint branches are not excluded.

2. Along this branch, for every $b \geq 1/5$,

$$J_b''(u) > \frac{21}{400} \quad (u \in [0, 1]). \quad (21)$$

Each J_b therefore has a unique minimizer u_b^ on $[0, 1]$.*

3. At $u_0 = 1/4$, the two original cash states satisfy the strict bounds

$$\frac{149}{1000} < J'_{1/5}(1/4) < \frac{1491}{10000}, \quad (22)$$

$$-\frac{103}{2500} < J'_4(1/4) < -\frac{41}{1000}. \quad (23)$$

Consequently,

$$u_{1/5}^* < \frac{1}{4} < u_4^*. \quad (24)$$

The minimizers are permitted to lie at allocation endpoints.

4. The observations in (17) are identical for $b = 1/5$ and $b = 4$. No rule in Definition 2 chooses an allocation that is optimal in both economies on this branch.

Proof structure. Cash enters the counterparty’s private certainty equivalent as an additive constant. The unconstrained first-order conditions therefore produce the same affine map at both cash levels. Uniform positive cash and position margins show that these unconstrained choices are actual constrained optima for every allocation, including the endpoints. The second step proves a curvature bound for the sum of all three shortfalls, allowing the counterparty’s own curvature contribution to be negative. Rational enclosures of normal integrals then establish the two derivative signs at $1/4$. Strict convexity separates the unique minimizers, and identical rule inputs cannot select both. Appendices A and B supply every step.

The counterparty takes more additional risk as u increases: $R'(u) = 50/101 > 0$. The price and this response are identical across cash states. The difference lies in which terminal payoff states leave the counterparty short of funds. The theorem shows that this difference is large enough in the stated construction to change the allocation minimizing the original objective. A change in risk levels alone would not establish that conclusion.

The result gives an existence claim about the precise observation set, not about all available financial information. Adding counterparty cash or an equivalent statistic removes the particular observational equivalence. Nor does the theorem say that every cash shift reverses a priority, that these optima are optimal across other equilibria, or that the model describes a complete default-recovery policy.

3.1 Displaying the original construction

Figure 1 shows the common price and aggregate-trade maps. The curves overlap because of the exact formulas in Theorem 1, not because of a fitted or approximate match. Figure 2 compares the two funding objectives after subtracting each objective’s value at $u = 1/4$. This normalization removes the level difference due to cash without changing either minimizer. The derivative panel makes their opposite local directions at that allocation visible.

The figure script evaluates the already derived functions. It checks derivatives at both allocation endpoints before using a bracketed derivative root for an interior minimum. Exact numerical locations are not needed for the theorem. Appendix H provides the data and reproduction entry points.

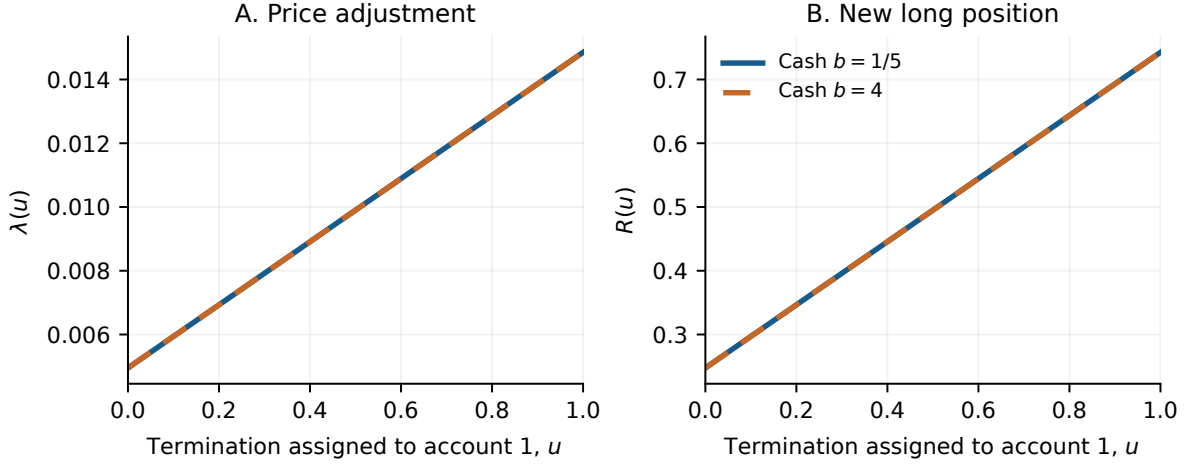


Figure 1: Identical price and trading maps. Both curves in each panel use Table 1 and the full original allocation interval. The blue solid and orange dashed curves correspond to $b = 1/5$ and $b = 4$ and coincide exactly. Cash-margin slack is not plotted and is not identical.

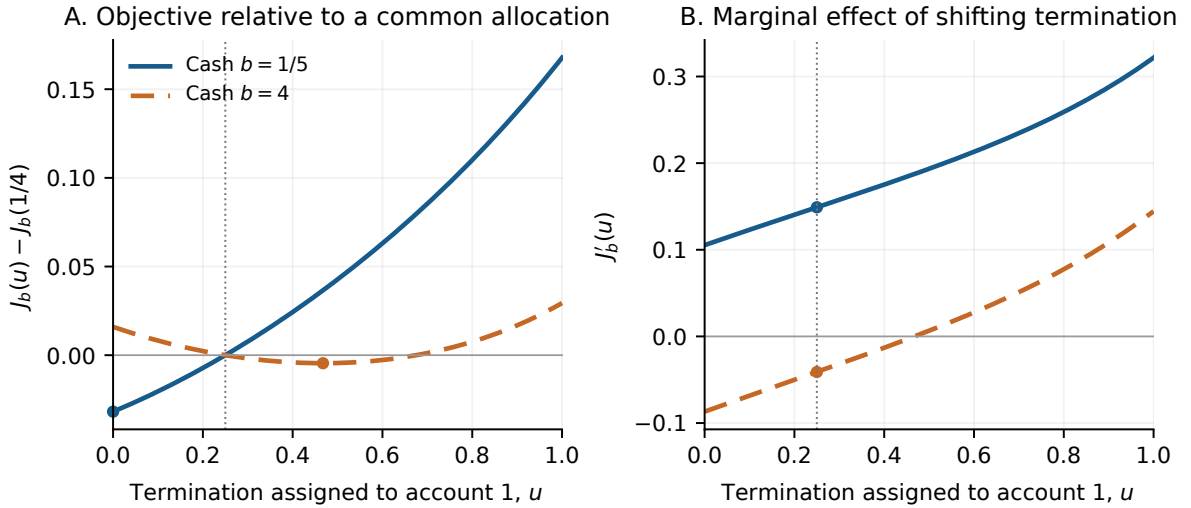


Figure 2: Different allocation objectives under the common map. Panel A plots $J_b(u) - J_b(1/4)$; dots mark numerical minimizing locations, including an endpoint when selected by the derivative test. Panel B plots $J'_b(u)$, with dots at $u = 1/4$. The dotted vertical line marks that comparison allocation. These are display calculations for the original parameters; the full-domain convexity and sign claims are proved in Appendices A–B.

4 When Does Counterparty Cash Change Priorities?

Theorem 1 establishes different optimal allocations despite identical trading maps. This section identifies the channel behind that construction and the cases in which it does not change a local comparison. A cash buffer determines which payoff states require replenishment. What matters for allocation is how a small transfer of the termination task changes account resources in those states.

4.1 A common map and the states with a shortfall

Fix a relative interior allocation d_0 and a direction $\delta \in T = \text{span}(\mathcal{D} - d_0)$ as defined in Section 2.5. All derivatives below are along the fixed-total allocation domain. Write $\theta_b = (s, b)$ and retain the observation in (17): known structure and the full specified trading map, excluding counterparty cash and equivalent quantities.

For the mechanism results, normality is not required. Instead, we state precisely the properties of the common map and resulting resources that are needed. They hold in the normal construction and can also hold under other payoff distributions. A different payoff distribution requires its own valid private equilibrium; the normal certainty equivalents are not imposed outside the normal model.

Lemma 1 (Cash-invariant behavior on a feasible branch). *Suppose, on a common allocation domain, that private unconstrained optima at given prices are unique and do not depend on b . Suppose a specified smooth solution of their market-clearing equations exists and is strictly feasible for every cash state being compared. Then this solution is an equilibrium for every such state. Its price, trade and position map, and all existing derivatives of that map, are independent of b .*

Cash entering a certainty equivalent as an additive constant is one way to satisfy the first condition. The final condition is substantive: unchanged unconstrained demand is not enough if a lower buffer makes those trades infeasible. Theorem 1 verifies it uniformly in the original construction. The lemma does not identify the numerical margin slack or eliminate other equilibria.

On this common map, let the counterparty's final resources and the other accounts' aggregate objective be

$$e_D(d; s, b, \omega) = b + V(d; s, \omega), \quad J(d; s, b) = J_H(d; s) + \mathbb{E}[(-b - V(d; s))^+]. \quad (25)$$

Both V and J_H are independent of b . The resource V includes equilibrium price changes, trades and real execution costs. In the original model, $V = \lambda R - k_D R^2/2 + z_D X$. Thus the decomposition keeps the entire counterparty response, not only a fixed-price demand derivative.

Proposition 1 (Cash and the value of a local allocation derivative). *Fix s, d_0, δ and a cash interval $I = [b_L, b_H]$. Suppose Lemma 1 applies, J_H is finite and differentiable near d_0 , and:*

1. $V(d)$ is almost surely continuously differentiable in a relative neighborhood of d_0 , $\mathbb{E}|V(d_0)| < \infty$, and its first derivatives there admit a common integrable bound;
2. $\Pr(V(d_0) = -b) = 0$ for every $b \in I$.

Put $a_\delta = D_\delta J_H(d_0; s)$ and $Y_\delta = D_\delta V(d_0; s)$, with V without an argument denoting $V(d_0; s)$. Then

$$g_\delta(b) := D_\delta J(d_0; s, b) = a_\delta - \mathbb{E}[Y_\delta \mathbf{1}\{V < -b\}], \quad (26)$$

$$g_\delta(b_2) - g_\delta(b_1) = \mathbb{E}[Y_\delta \mathbf{1}\{-b_2 \leq V < -b_1\}] \quad (b_1 < b_2). \quad (27)$$

If the law of V is absolutely continuous on the swept interval, with density f_V , let $m_\delta(t) = \mathbb{E}[Y_\delta \mid V = t]$ be a measurable version. Then g_δ is absolutely continuous on I and

$$g'_\delta(b) = m_\delta(-b)f_V(-b) \quad \text{for almost every } b \in I. \quad (28)$$

The allowed observation identifies the value of g_δ within this cash-information class if and only if

$$m_\delta(t)f_V(t) = 0 \quad \text{for almost every } t \in (-b_H, -b_L). \quad (29)$$

The proof is in Appendix C. Increasing cash removes a band of payoff states from the shortfall region. Equation (27) measures the average resources generated by the allocation adjustment in precisely that band. This gives three ways for cash to leave the derivative unchanged: the adjustment does not change counterparty resources; the band has no probability mass; or its conditional marginal resources cancel.

The proposition characterizes a derivative within the specified information class. A derivative can change in size without changing sign, and different feasible gradients can have a common descent direction. The separate construction in Theorem 1 establishes the stronger conclusion of incompatible optimal allocations.

4.2 When the local direction reverses

For a one-factor counterparty account, write

$$V(d) = \kappa(d) + z(d)X, \quad z(d_0) > 0, \quad t(b) = -\frac{b + \kappa}{z}. \quad (30)$$

Here and below κ, z and their derivatives are evaluated at d_0 . The cutoff $t(b)$ is the payoff below which the counterparty needs funds. Assume $\mathbb{E}|X| < \infty$ and a continuous density f_X ; let F_X be its CDF and $L_X(t) = \mathbb{E}[X\mathbf{1}\{X < t\}]$. With $\kappa_\delta = D_\delta\kappa$ and $z_\delta = D_\delta z$, the counterparty's contribution is

$$C_\delta(b) = -\kappa_\delta F_X(t(b)) - z_\delta L_X(t(b)), \quad g_\delta(b) = a_\delta + C_\delta(b), \quad (31)$$

$$g'_\delta(b) = \frac{\kappa_\delta + z_\delta t(b)}{z} f_X(t(b)). \quad (32)$$

Proposition 2 (A threshold for a fixed local priority). *Retain the differentiability and common-map conditions above for all $b \geq b_L$. Suppose $z_\delta > 0$, f_X is strictly positive on $(-\infty, t(b_L)]$, and*

$$t(b_L) < -\frac{\kappa_\delta}{z_\delta}. \quad (33)$$

Then $C_\delta(b)$ is positive, continuous and strictly decreasing to zero, so $g_\delta(b)$ is strictly decreasing to a_δ .

1. *If $a_\delta \geq 0$, then $g_\delta(b) > 0$ at every finite cash level.*
2. *If $a_\delta < 0$ and $g_\delta(b_L) \leq 0$, then $g_\delta(b) < 0$ for all $b > b_L$.*
3. *If $a_\delta < 0 < g_\delta(b_L)$, there is a unique finite $b^*(d_0, \delta; s) > b_L$ with $g_\delta(b^*) = 0$. The derivative is positive below this threshold and negative above it.*

Under these conditions, a strict reversal within a finite comparison interval occurs exactly when its endpoint derivatives are strictly opposite in sign. At a zero derivative, there is no strict first-order direction to report. The threshold is tied to the chosen allocation, direction and structure;

it is not a threshold for K or a claim that the globally optimal allocation moves monotonically with cash.

The tail condition has a direct meaning. In every state where the counterparty is short of funds, the proposed increase in its exposure reduces its resources. Its marginal shortfall cost therefore opposes an adjustment that benefits the other accounts. Low cash can make this cost dominate; more cash eventually removes enough loss states for the other accounts' benefit to dominate. If the two sides favor the same direction, or the counterparty effect is too small to change the sign, cash does not reverse the priority.

The model's existing margin requirement can make the tail condition hold. On a regular branch with counterparty risk penalty $\rho_D(A + R - H_D)$ and the same quadratic cost, let $\nu = (k_D + \rho_D'')^{-1} > 0$. For a direction with $R_\delta = \nu\lambda_\delta > 0$,

$$\frac{\kappa_\delta}{z_\delta} = \lambda + R\rho_D'', \quad t(b) \leq -(m_D + \lambda). \quad (34)$$

Thus $m_D > R\rho_D''$ is sufficient for (33). In the normal model $\rho_D'' = \alpha_D$. Appendix C.4 derives the link without adding a margin rule.

At the construction's $u_0 = 1/4$, $z_\delta = 50/101 > 0$, $R = 75/202$ and $m_D = 1/10 > \alpha_D R$. The original two derivative signs therefore lie on opposite sides of a unique local cash threshold in $(1/5, 4)$. Table 2 separates the hedgers' and counterparty's contributions, using the same original parameters. Its displayed decimals illustrate the mechanism; the theorem's sign evidence remains the exact rational certificate.

Table 2: The same local trading response, different funding priorities

Quantity at $u = 1/4$	$b = 1/5$	$b = 4$
Price response λ'	0.009901	0.009901
Counterparty trade response R'	0.495050	0.495050
Hedgers' contribution a_δ	-0.043883	-0.043883
Counterparty contribution C_δ	0.192929	0.002783
Total derivative g_δ	0.149046	-0.041100
Strict improving move	Lower u	Raise u

4.3 What additional state information is sufficient?

Corollary 1 (The shortfall cutoff as a state input). *In (31), suppose the structure, payoff distribution and common map are known. Observing the counterparty's cutoff t at d_0 suffices to calculate every feasible local derivative there. If F_X is strictly increasing on the relevant region, its account shortfall probability $\pi_D = F_X(t)$ is also sufficient through $t = F_X^{-1}(\pi_D)$.*

The cutoff supplies $F_X(t)$ and the tail first moment $L_X(t)$ in (31). Under the stated knowledge it also reveals the missing cash, $b = -zt - \kappa$. A single observed cutoff can therefore recover the fixed cash state used to reconstruct the objective elsewhere on the common domain. A sign-only input indicating which side of a *local* threshold cash lies on does not generally do this.

This is an economically meaningful description of the missing state, not a minimum-information-dimension theorem. A probability alone is insufficient to recover the tail moment when the payoff distribution is unknown. The result neither reduces the raw information needed to estimate the map and distribution nor asserts that a clearing institution lacks the underlying cash data.

5 A Separable Benchmark

Re-trading does not always make the counterparty's cash relevant for choosing an allocation. A useful benchmark arises when the two hedgers leave the same fraction of an additional termination in place at a fixed price. Aggregate rebuilding then depends on the total task, not on who receives it. This section returns to the normal model in Section 2.

5.1 Retention and price invariance

Recall $\alpha_i = \gamma_i \sigma^2$. Define

$$\eta_i = \alpha_i + k_i, \quad v_i = \eta_i^{-1}, \quad \beta_i = \alpha_i / \eta_i, \quad \tau_i = k_i / \eta_i = 1 - \beta_i. \quad (35)$$

When hedger constraints are slack, private demand is $r_i = \beta_i(H_i - q_i + d_i) - v_i \lambda$. At a fixed price, an additional unit of termination induces β_i units of rebuilding and leaves τ_i units of position reduction. The retention share is an account's fixed-price response, not a measure of total system risk removed.

For a specified smooth counterparty branch, let $R_D(\lambda)$ be its response and $\nu = R'_D(\lambda)$ its slope. Set $S = v_1 + v_2$. A prime in the next lemma denotes a small transfer from account j to account i .

Lemma 2 (Price invariance for a fixed-total transfer). *For a feasible transfer between distinct accounts i and j , suppose the hedgers' cash and position constraints are strictly slack, final positions are positive, the selected equilibrium branch is smooth, and $S + \nu$ is finite and nonzero. Then*

$$\lambda' = \frac{\beta_i - \beta_j}{S + \nu} = \frac{\tau_j - \tau_i}{S + \nu}. \quad (36)$$

The price is unchanged to first order if and only if the retention shares match.

Unequal retention alone does not determine the sign of the objective derivative. It identifies a price response, which still has to be evaluated together with all accounts' cash and position changes.

5.2 Allocation with equal retention

Proposition 3 (Separation and marginal-benefit equalization). *Suppose $\tau_1 = \tau_2 = \tau$, write $\beta = 1 - \tau$, and let $\alpha_i, k_i, \sigma > 0$. Suppose a nonempty compact convex comparison domain within $\mathcal{D}$ stays entirely on the regular branch of Definition 1, with $z_i > 0$. Then λ, R, c_D, z_D are constant over the domain, and*

$$J_{\text{reg}}(d; b) = f_1(d_1) + f_2(d_2) + F_\sigma(c_D, z_D), \quad f_i(d_i) = F_\sigma(c_i(d_i), z_i(d_i)) \quad (37)$$

is strictly convex and has a unique minimizer there. Put

$$\pi_i = \Phi\left(-\frac{c_i}{\sigma z_i}\right), \quad s_i = \sigma \phi\left(\frac{c_i}{\sigma z_i}\right), \quad G_i = \lambda + k_i r_i, \quad I_i^0 = -f'_i = \tau s_i - \beta \pi_i G_i. \quad (38)$$

For the original domain with only the fixed sum and position bounds, the minimum is characterized by a common multiplier μ_Q and

$$I_i^0 = \mu_Q \ (0 < d_i < q_i), \quad I_i^0 \leq \mu_Q \ (d_i = 0 < q_i), \quad I_i^0 \geq \mu_Q \ (d_i = q_i > 0). \quad (39)$$

Coordinates fixed by coincident bounds are omitted from these comparisons. Additional domain restrictions require their own boundary conditions.

The first term in I_i^0 is the benefit from lowering exposure that remains after rebuilding. The second is the associated change in the terminal cash intercept. Because G_i may have either sign, rebuilding is not assumed to destroy cash in every case. Own marginal benefits decline strictly, so equalization describes priorities that update as allocations change, rather than a single fixed queue.

Under equal retention, the counterparty's whole statewise resource V is allocation-invariant. Thus $Y_\delta = 0$ in Proposition 1, and its shortfall contributes a constant to every allocation comparison. Cash may still change the risk level. This connects a transparent positive allocation result to the general zero-effect condition; equal retention is not necessary for every case of that condition.

The benchmark concerns the stated regular domain and the expected-shortfall criterion. It does not establish the distribution-free, statewise or path-independence properties of static ADL rules. Nor does its convexity argument apply to the heterogeneous-retention construction in Theorem 1, which has its own proof. Appendix D contains the derivation.

When retention differs, Appendix D.3 gives the existing complete local accounting formula $D_{ij} = I_j - I_i$, with $I_i = \tau_i(s_i + \Lambda) - \beta_i \pi_i G_i$. The common term Λ includes all participants' price and cash effects. It is a computable representation conditional on the branch and required states, not a minimum-dimensional statistic or an outer multiplier on Q .

6 Robustness and Economic Scope

6.1 A class of nonnormal payoff distributions

The common-map construction is not confined to a single Gaussian distribution. The following result retains all original positions, cash states, cost coefficients, margin requirements and settlement flows. It varies the payoff distribution within a specific small perturbation and uses the correct CARA certainty equivalents for that distribution.

Let

$$P_\varepsilon = (1 - \varepsilon)N(0, 1) + \frac{\varepsilon}{2}N(0, 1/2) + \frac{\varepsilon}{2}N(0, 3/2), \quad 0 \leq \varepsilon \leq 1, \quad (40)$$

where the second normal parameter is the variance. These laws have mean zero and variance one, and fourth moment $3 + 3\varepsilon/4$. Thus each positive ε gives a nonnormal law with the same payoff variance as the original construction.

Proposition 4 (Persistence under small nonnormal perturbations). *For the original parameters and cash pair $\{1/5, 4\}$, there is an $\varepsilon_0 \in (0, 1]$ such that, for every $0 < \varepsilon < \varepsilon_0$:*

1. *the original regular branch has a smooth continuation on all $u \in [0, 1]$, with every trading constraint strictly slack in both cash states;*
2. *at this fixed ε , the two cash states share the complete price, trade and position map, and all its existing derivatives;*
3. *each branch objective satisfies $J''_{\varepsilon,b} > 21/800$ and has a unique minimizer, with*

$$u_{\varepsilon,1/5}^* < 1/4 < u_{\varepsilon,4}^*. \quad (41)$$

Consequently, the same observation-based rule cannot choose a branch-optimal allocation in both cash states or a strict first-order improvement at $u = 1/4$ in both.

Appendix E derives the nonnormal demand equations and gives the continuation proof. Strict concavity and a uniformly nonzero clearing derivative continue the original map. The original slack, curvature and derivative-sign margins preserve feasibility and the separated optima.

Different perturbations can produce different trading maps. Equality is across the two cash states *at a given distribution*. The result is an analytic existence statement: it does not give a numerical maximum ε_0 and is not supported by a simulation or parameter scan. It retains cash-translation-invariant CARA preferences and does not extend to arbitrary distributions or general wealth effects.

6.2 What the criterion and information result cover

The terminal funding interpretation in Section 2.4 gives the objective a concrete quantity: cash required from outside separately funded accounts to meet final contractual obligations. A policy can reduce that quantity even when no permanent default occurs. This does not make funding principal a resource cost or a welfare measure. Its relevance depends on the settlement deadline, access to counted collateral, availability of outside resources and the absence of a continuing margin floor at the modeled endpoint.

The private full-recourse payoffs include the economic consequences of terminal positions. They do not include an additional emergency financing fee generated by the institution's objective. If such a fee changed private choices, the demand equations and cash invariance would require new analysis. Likewise, replacing $(-e_j)^+$ by a continuing-margin restoration target $(h_j - e_j)^+$ generally changes allocation rankings. Neither change is part of the results here.

The institution's observation is deliberately defined. A real clearing institution may obtain cash or related information from its members; the theorem does not claim otherwise. The result identifies a limitation of using even a complete trading map *alone with the stated structure* to assess the chosen funding objective. Estimating that map and its behavioral parameters is a separate empirical problem.

Finally, all global allocation statements use their designated common regular branch. Other constrained equilibria can coexist, constraints can change the response map, and branch changes can remove differentiability. The model neither selects among such equilibria nor describes a full default waterfall, repeated intervention or an actual episode of market contagion.

7 Conclusion

Complete knowledge of trading responses can leave a termination-allocation problem unresolved. In the construction studied here, the same price, trade and position map supports different unique minimizing allocations because counterparty cash changes the states in which funding is needed. The limitation concerns a specified observation set and a common regular branch under an expected account-funding-shortfall criterion.

The mechanism also describes when the omitted state does not matter. Marginal comparisons are unchanged when the counterparty's conditional resource response vanishes over the states moved across the shortfall boundary. Equal retention provides a concrete separable benchmark. Under the additional tail and crossing conditions, cash instead reverses a fixed local priority; a known-distribution shortfall cutoff identifies the missing state. Small nonnormal perturbations preserve the original allocation construction.

The practical distinction is between predicting trades and evaluating their funding consequences. Applying it to a clearing procedure requires matching the criterion, settlement resources, available information and permissible intervention to the institution's actual decision.

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A Complete Proof of Theorem 1

A.1 Private optima and the common map

Put $\eta_j = \alpha_j + k_j > 0$, $v_i = \eta_i^{-1}$ and $\beta_i = \alpha_i/\eta_i$. At a fixed price, differentiating (11)–(12) yields

$$r_i = \beta_i(H_i - a_i) - v_i\lambda, \quad (42)$$

$$R = \frac{\lambda - \alpha_D(A - H_D)}{\eta_D}. \quad (43)$$

The second derivatives of the objectives are $-\eta_i$ and $-\eta_D$, so these are the unique unconstrained optima. They become the unique constrained optima whenever they satisfy the funding and position restrictions. Differentiating (42) at a fixed price gives $\partial r_i/\partial d_i = \beta_i$ and $\partial z_i/\partial d_i = -1 + \beta_i = -\tau_i$, which justifies the retention interpretation in Section 3.

Under Table 1, $\eta_1 = \eta_2 = 4$, $\eta_D = 1/50$ and $A = H_D = 1$. Hence

$$r_1 = \frac{3}{4}u - \frac{1}{4}\lambda, \quad r_2 = \frac{1}{4}(1 - u) - \frac{1}{4}\lambda, \quad R = 50\lambda. \quad (44)$$

Market clearing reduces to

$$\frac{1}{4} + \frac{1}{2}u - \frac{1}{2}\lambda = 50\lambda.$$

It has the unique solution for λ in (19); substitution gives all the remaining expressions. Cash b does not enter this linear system. Every fully slack equilibrium must solve the same system, so at most one such equilibrium exists. Its actual feasibility is established next.

A.2 Uniform feasibility on the entire allocation interval

The affine expressions in (19)–(20) give, for every $u \in [0, 1]$,

$$0 < \lambda \leq \frac{3}{202}, \quad |r_1| < \frac{3}{4}, \quad |r_2| < \frac{1}{4}, \quad 0 < R < \frac{3}{4}, \quad (45)$$

$$\frac{603}{808} \leq z_1 < 1, \quad \frac{201}{808} \leq z_2 < 1, \quad \frac{126}{101} \leq z_D < \frac{7}{4}. \quad (46)$$

For example, an affine function attains its extrema at the interval endpoints, so these bounds follow from its exact endpoint values. This is an interval argument for an affine function, not numerical sampling of equilibria. Negative r_i near an endpoint is allowed by (6).

Since $a_i\lambda \geq 0$, lower bounds for cash minus required margin follow directly from (45)–(46):

$$M_1 - m_1z_1 > \frac{1}{2} - \frac{9}{32} - \frac{1}{10} = \frac{19}{160} > 0, \quad (47)$$

$$M_2 - m_2z_2 > 3 - \frac{3}{32} - \frac{1}{10} = \frac{449}{160} > 0, \quad (48)$$

$$M_D - m_Dz_D > \frac{1}{5} - \frac{3}{202} - \frac{9}{3200} - \frac{7}{40} = \frac{2371}{323200} > 0, \quad (49)$$

$$K - R > \frac{1}{4} > 0. \quad (50)$$

For the third inequality we first set $b = 1/5$; higher b only increases the left side. The positions are strictly positive by (46). Thus all the unconstrained optima are strictly feasible on the full domain and are also the constrained optima. Market clearing has already been imposed. This proves part 1 of the theorem without making a claim about constrained equilibria.

A.3 Shortfall derivatives and uniform curvature

For this construction $\sigma = 1$; abbreviate F_1 to F . If $X \sim N(0, 1)$ and $z > 0$, the short-account expectation is

$$\mathbb{E}[(zX - c)^+] = \int_{c/z}^{\infty} (zx - c)\phi(x) dx = z\phi(c/z) - c\Phi(-c/z).$$

The last equality uses $\phi'(x) = -x\phi(x)$. Symmetry of X gives the same formula for a long account. Direct differentiation gives

$$F_c = -\Phi(-c/z), \quad F_z = \phi(c/z), \quad \nabla^2 F = \frac{\phi(c/z)}{z} \begin{pmatrix} 1 & -c/z \\ -c/z & (c/z)^2 \end{pmatrix}. \quad (51)$$

The Hessian is positive semidefinite. Define $\pi_j = \Phi(-c_j/z_j)$; this is a funding-shortfall probability. All positions are affine in u , and therefore

$$\frac{d^2}{du^2} F(c_j(u), z_j(u)) = \frac{\phi(c_j/z_j)}{z_j} \left(c'_j - \frac{c_j}{z_j} z'_j \right)^2 - \pi_j c''_j. \quad (52)$$

No assumption that the entire counterparty contribution is convex is needed.

The slopes of the affine map are

$$\lambda' = \frac{1}{101}, \quad r'_1 = \frac{151}{202}, \quad r'_2 = -\frac{51}{202}, \quad R' = \frac{50}{101}, \quad z'_1 = -\frac{51}{202}, \quad z'_2 = \frac{151}{202}, \quad z'_D = \frac{50}{101}. \quad (53)$$

Because $c_i = E_i - \lambda r_i - k_i r_i^2/2$ and $c_D = b + \lambda R - k_D R^2/2$, their second derivatives are constant:

$$c''_1 = -2\lambda' r'_1 - k_1 (r'_1)^2 = -\frac{23405}{40804} < -\frac{1}{2}, \quad (54)$$

$$c''_2 = -2\lambda' r'_2 - k_2 (r'_2)^2 = -\frac{7599}{40804} < 0, \quad (55)$$

$$c''_D = 2\lambda' R' - k_D (R')^2 = \frac{75}{10201} < \frac{1}{100}. \quad (56)$$

Completing the square in $-\lambda r_1 - r_1^2/2$ gives

$$c_1 \leq \frac{1}{2} + \frac{\lambda^2}{2} \leq \frac{1}{2} + \frac{1}{2} \left(\frac{3}{202} \right)^2, \quad \frac{c_1}{z_1} < 1, \quad (57)$$

where the strict ratio bound follows on using $z_1 \geq 603/808$. Hence $\pi_1 > \Phi(-1) > 1/8$. Appendix B.2 proves the last numerical inequality by an elementary integral bound.

Drop the nonnegative square terms in (52) and the positive term $-\pi_2 c''_2$. Since $0 \leq \pi_D \leq 1$, we obtain, uniformly for $b \geq 1/5$ and $u \in [0, 1]$,

$$J''_b(u) > \frac{1}{8} \frac{1}{2} - \frac{1}{100} = \frac{21}{400}. \quad (58)$$

The positive positions ensure that the composed objective is smooth on a neighborhood of $[0, 1]$. Its continuity gives a minimum on this compact interval, and strict convexity makes that minimum unique. This proves part 2 independently of any equal-retention result.

Table 3: Exact inputs for the derivative at $u_0 = 1/4$

Account	c	z	c'	z'
1	157141/326432	189/202	-1491/10201	-51/202
2	961971/326432	44/101	5739/40804	151/202
D	$b + 675/326432$	277/202	225/40804	50/101

A.4 Opposite signs at the common allocation

At $u_0 = 1/4$ the exact values are

$$\lambda = \frac{3}{404}, \quad r_1 = r_2 = \frac{75}{404}, \quad R = \frac{75}{202}. \quad (59)$$

Combining these with (53) gives Table 3. In particular, the hedger inputs do not depend on b . Equation (51) yields the exact expression

$$J'_b(1/4) = \sum_{j \in \{1,2,D\}} [-c'_j \Phi(-c_j/z_j) + z'_j \phi(c_j/z_j)]. \quad (60)$$

Appendix B supplies finite rational upper and lower bounds for each normal quantity in this expression. Applying those bounds to the exact inputs in Table 3 gives

Counterparty cash	Strict lower bound for $J'_b(1/4)$	Strict upper bound
1/5	149/1000	1491/10000
4	-103/2500	-41/1000

These are outward rational enclosures, not signs inferred from displayed floating-point values. The finite sums and remainder bounds, including the normalization of the normal density, are specified in Appendix B so that the sign step can be checked independently of the implementation.

A.5 Separated minima and the rule conclusion

For a differentiable strictly convex function on $[0, 1]$, the derivative is strictly increasing here because of (58). When $J'_{1/5}(1/4) > 0$, every point to the right has a larger derivative; a small move left from $1/4$ lowers the objective. Its unique minimum must therefore be strictly to the left. The same argument with signs reversed puts the unique minimum of J_4 strictly to the right. Either minimum can be an endpoint. This proves (24) and part 3.

All components of s and the entire map (16) coincide for the two states. Hence $\mathcal{O}(\theta_{1/5}) = \mathcal{O}(\theta_4)$. A deterministic rule must return the same u for that same input. The two singleton argmin sets are disjoint by (24), so no such rule satisfies (18) for both states. This proves part 4 and completes the theorem. $\square$

For completeness, at $u_0 = 1/4$ a scalar move Δu is a strict first-order improvement only if $\Delta u J'_b(1/4) < 0$. The two signs require opposite moves. This local implication is distinct from the global-minimizer conclusion and uses no assertion that unequal derivative values alone imply incompatible choices.

B Rational Normal-Integral Bounds

This appendix supplies the numerical inequalities used in the proof as a finite rational certificate. It evaluates only the parameters in Table 1 and the point $u_0 = 1/4$. It does not locate either optimum, sample allocations to establish feasibility, or search for other equilibria.

B.1 Normalizing the density

For $0 < x < 1$ let

$$A_{80}(x) = \sum_{k=0}^{80} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad A_-(x) = A_{80}(x) - \frac{x^{163}}{163}, \quad A_+(x) = A_{80}(x). \quad (61)$$

The alternating arctangent expansion implies $A_-(x) < \arctan x < A_+(x)$. The tangent addition formula, with the angles chosen in the first quadrant, gives Machin's identity $\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$. Therefore the following are rational lower and upper bounds for π :

$$\begin{aligned} p_- &= 16A_-(1/5) - 4A_+(1/239), \\ p_+ &= 16A_+(1/5) - 4A_-(1/239). \end{aligned} \quad (62)$$

Set

$$C_- = \frac{3989422804014326}{10^{16}}, \quad C_+ = \frac{3989422804014328}{10^{16}}. \quad (63)$$

Substitution of the finite sums in (62), using rational arithmetic, gives

$$2p_+C_-^2 < 1 < 2p_-C_+^2. \quad (64)$$

It follows that $C_- < \phi(0) = (2\pi)^{-1/2} < C_+$. Thus no floating-point normal-library value is an input to the sign argument.

B.2 The tail bound used for curvature

Since $C_+ < 2/5$ and $e^{-t^2/2} \leq 1 - t^2/2 + t^4/8$ for $0 \leq t \leq 1$ by Taylor's remainder,

$$\begin{aligned} \Phi(-1) &= \frac{1}{2} - \phi(0) \int_0^1 e^{-t^2/2} dt \\ &> \frac{1}{2} - \frac{2}{5} \left(1 - \frac{1}{6} + \frac{1}{40} \right) = \frac{47}{300} > \frac{1}{8}. \end{aligned} \quad (65)$$

This gives a readable analytic bound for Appendix A.3; the supplied certificate also checks the weaker $1/8$ bound using its normal-integral enclosure.

B.3 Finite bounds at the comparison point

For a positive rational w define $h = w^2/2$ and

$$S_{80}(h) = \sum_{k=0}^{80} \frac{(-h)^k}{k!}, \quad S_-(h) = S_{80}(h) - \frac{h^{81}}{81!}, \quad S_+(h) = S_{80}(h), \quad (66)$$

$$\begin{aligned} T_{80}(w) &= \sum_{k=0}^{80} \frac{(-1)^k w^{2k+1}}{2^k k! (2k+1)}, \\ T_-(w) &= T_{80}(w) - \frac{w^{163}}{2^{81} 81! 163}, \quad T_+(w) = T_{80}(w). \end{aligned} \quad (67)$$

Even and odd Taylor sums bound e^{-h} . Integrating the corresponding Taylor bounds for $e^{-t^2/2}$ from 0 to w gives

$$S_-(h) \leq e^{-h} \leq S_+(h), \quad T_-(w) \leq \int_0^w e^{-t^2/2} dt \leq T_+(w). \quad (68)$$

Taylor's remainder fixes the sign of each bound. One need not assume that the early terms are decreasing. All lower sums S_- and T_- at the arguments used below are positive, as verified by substituting these rational arguments into the finite sums.

Combining (63) and (68), intervals enclosing $\phi(w)$ and $\Phi(-w)$ are

$$\mathcal{P}(w) = [C_- S_-(w^2/2), C_+ S_+(w^2/2)], \quad (69)$$

$$\mathcal{N}(w) = [1/2 - C_+ T_+(w), 1/2 - C_- T_-(w)]. \quad (70)$$

The four arguments are $w_1 = c_1/z_1$, $w_2 = c_2/z_2$ and the two values $w_D(b) = c_D/z_D$ in Table 3. Explicitly,

$$w_1 = \frac{157141}{305424}, \quad w_2 = \frac{961971}{142208}, \quad w_D(b) = \frac{b + 675/326432}{277/202}, \quad b \in \{1/5, 4\}. \quad (71)$$

For an interval $I = [l, u]$ and a rational coefficient a , use

$$aI = \begin{cases} [al, au], & a \geq 0, \\ [au, al], & a < 0, \end{cases} \quad [l, u] + [l', u'] = [l + l', u + u']. \quad (72)$$

With the exact c'_j, z'_j from Table 3, define the entirely rational interval

$$\mathcal{G}(b) = \sum_{j \in \{1, 2, D\}} [-c'_j \mathcal{N}(w_j) + z'_j \mathcal{P}(w_j)]. \quad (73)$$

Equation (60) belongs to this interval. Direct finite-sum evaluation gives the rational containments

$$\mathcal{G}(1/5) \subset \left(\frac{149}{1000}, \frac{1491}{10000} \right), \quad (74)$$

$$\mathcal{G}(4) \subset \left(-\frac{103}{2500}, -\frac{41}{1000} \right). \quad (75)$$

All arithmetic required to verify these containments consists of the finite sums (61), (66), (67), the rational input table, and the signed interval rules (72). Endpoints are rounded outward only in the two displayed final containments.

B.4 Reproduction and the role of the certificate

The accompanying `replication/original_witness/` directory contains the unchanged input parameters and the supplied `certify_original_witness.py`. It uses Python's standard-library `Fraction` class to reproduce the affine identities, feasibility bounds, curvature bound and the two sign enclosures. Its Gaussian CDF calculation is the equivalent form of (70) obtained by reflecting $\Phi(w)$ to $\Phi(-w)$. The recorded run matches the supplied baseline.

The separate `verify_paper_certificate.py` evaluates the exact input table and the finite-sum interval formulas as written in this appendix. It is a check that the new exposition agrees with the original certificate, not an additional model exercise. The scripts retain their outputs and exit status. The mathematical argument for why the bounds hold and why the signs separate unique minimizers is given above; neither a numerical graph nor a solver's reported optimizer replaces it.

C Proofs for the Cash Mechanism

C.1 Cash-invariant behavior and observation classes

Proof of Lemma 1. By assumption b changes neither the unconstrained private choices at given prices nor the market-clearing equations. The same selected solution therefore applies to each cash state. Each unconstrained maximizer is strictly feasible, so it also maximizes over the constrained set. This establishes equilibrium for each state. Equality of maps gives equality of all derivatives that exist. Numerical slack and other constraint-branch solutions need not coincide. $\square$

At a relative interior d_0 , let $T = \text{span}(\mathcal{D} - d_0)$ and let P_T be its orthogonal projection. Define $G(\theta) = P_T \nabla_d J(d_0; \theta)$; projecting any differentiable extension off the affine hull gives the same G . For one observation o , let $\Theta_o = \{\theta : \mathcal{O}(\theta) = o\}$. A rule can return the correct feasible gradient throughout Θ_o exactly when G is constant there. It can select a strict first-order improvement whenever one exists exactly when

$$\bigcap_{\theta \in \Theta_o: G(\theta) \neq 0} \{\delta \in T : G(\theta)^\top \delta < 0\} \neq \emptyset. \quad (76)$$

If there are no nonzero-gradient states, that improvement requirement is vacuous. Likewise, provided every objective attains its minimum on the common domain, a common optimal allocation exists exactly when

$$\bigcap_{\theta \in \Theta_o} \arg \min_{d \in \mathcal{D}} J(d; \theta) \neq \emptyset. \quad (77)$$

Necessity follows from the same-input, same-output property. Sufficiency follows by selecting a common object in the relevant intersection. These are setwise existence statements, not additional claims about measurable or continuous selection. Theorem 1 supplies disjoint singleton argmins; Proposition 1 characterizes constancy of a derivative in a cash-information class.

C.2 Differentiation and the zero-effect condition

Proof of Proposition 1. Restrict d to $d_0 + h\delta$ for sufficiently small $|h|$. The positive-part function is 1-Lipschitz. The difference quotients of $(-b - V(d))^+$ are bounded in absolute value by the integrable local derivative envelope times $\|\delta\|$. Away from $V(d_0) = -b$, the chain rule gives their pointwise limit as $-Y_\delta \mathbf{1}\{V < -b\}$. The no-atom condition makes this valid almost surely. Dominated convergence proves (26); finiteness follows from integrability of $V(d_0)$ and its local envelope.

Subtract (26) at two cash levels. The difference of indicators selects $-b_2 \leq V < -b_1$; their boundaries have zero probability. This proves (27).

Conditional expectation and absolute continuity of the law on the swept interval give

$$g_\delta(b_2) - g_\delta(b_1) = \int_{-b_2}^{-b_1} m_\delta(t) f_V(t) dt. \quad (78)$$

Its integrand is integrable because $\int |m_\delta(t)| f_V(t) dt \leq \mathbb{E}|Y_\delta| < \infty$ on that interval. Hence g_δ is absolutely continuous. Differentiating the integral with its moving lower endpoint yields the positive sign in (28). An absolutely continuous function is constant on an interval exactly when its derivative vanishes almost everywhere. Since observations are identical throughout the cash class, constancy is exactly identification of its derivative value, proving (29).

For the complete feasible gradient, apply the same argument to a basis of T . Equivalently, use $Y = P_T \nabla_d V$ and require $\mathbb{E}[Y \mid V = t] f_V(t) = 0$ almost everywhere on the swept interval. This does not assert a necessary condition for a common optimum: that is the different intersection in (77). $\square$

C.3 One-factor threshold and state statistic

Substituting $V = \kappa + zX$ with $z > 0$ into (26) gives (31). Because $t'(b) = -1/z$, the fundamental theorem of calculus, $F'_X = f_X$ and $L'_X = t f_X$ give (32).

Proof of Proposition 2. For every shortfall state, $X < t(b) \leq t(b_L) < -\kappa_\delta/z_\delta$. Thus $\kappa_\delta + z_\delta X < 0$. Strictly positive tail density gives positive probability to such states, so $C_\delta = -\mathbb{E}[(\kappa_\delta + z_\delta X) \mathbf{1}\{X < t(b)\}] > 0$. Equation (32) is strictly negative at every finite b . It also establishes continuity under the density assumption.

As b tends to infinity, $t(b)$ tends to minus infinity and the indicator tends pointwise to zero. The integrable bound $|\kappa_\delta| + |z_\delta| |X|$ permits dominated convergence, giving $C_\delta(b) \rightarrow 0$. Consequently $g_\delta(b) \downarrow a_\delta$.

If $a_\delta \geq 0$, positive C_δ gives part 1. If $a_\delta < 0$ and the derivative starts nonpositive, strict decrease gives part 2, including the stated zero-at-the-endpoint case. Otherwise the initial derivative is positive and its limit is negative. Continuity gives a finite zero, and strict decrease gives uniqueness and the sign pattern in part 3. On a finite interval the same monotonicity makes strict endpoint sign opposition necessary and sufficient for a strict reversal within it. In the two-account coordinate, positive and negative derivatives require opposite directions for strict local improvement. $\square$

Proof of Corollary 1. Known t and the payoff distribution determine both $F_X(t)$ and $L_X(t)$. The known map determines κ_δ, z_δ , and the other accounts determine a_δ . Equation (31) then computes each direction. If F_X is strictly increasing, the shortfall probability uniquely recovers t on the relevant region. The identity $b = -zt - \kappa$ proves the stated cash recovery. No claim of reduced data collection follows from this reexpression. $\square$

C.4 The margin link

At a regular counterparty optimum with quadratic cost and a twice differentiable risk penalty,

$$\lambda = k_D R + \rho'_D (A + R - H_D), \quad \nu = \frac{1}{k_D + \rho''_D} > 0.$$

Since fixed Q fixes A , $z_\delta = R_\delta = \nu \lambda_\delta$. Differentiating $\kappa = \lambda R - k_D R^2/2$ gives

$$\kappa_\delta = R \lambda_\delta + (\lambda - k_D R) R_\delta.$$

Divide by $R_\delta > 0$ to obtain $\kappa_\delta/z_\delta = R/\nu + \lambda - k_D R = \lambda + R \rho''_D$. Meanwhile the unchanged cash constraint implies $b + \kappa \geq (m_D + \lambda)z$. These prove (34). The strict inequality $m_D > R \rho''_D$ implies the required strict cutoff bound, even when the budget itself only gives a weak inequality.

In Table 1 at $u_0 = 1/4$, $\rho''_D = \alpha_D = 1/100$ and $R = 75/202$, so the strict sufficient condition holds. Feasibility for all $b \geq 1/5$ is supplied by Theorem 1(1). Its opposite derivative signs and the monotonicity just proved establish a unique threshold between the original cash pair, without needing its numerical location.

D Price Invariance, Separation and Local Accounting

D.1 Price response

Proof of Lemma 2. Let the transfer vector satisfy $\delta_i = 1, \delta_j = -1$. Differentiating the slack hedger demands and the selected counterparty response gives

$$r'_\ell = \beta_\ell \delta_\ell - v_\ell \lambda', \quad z'_\ell = -\tau_\ell \delta_\ell - v_\ell \lambda', \quad R' = \nu \lambda'. \quad (79)$$

Market clearing implies $(S + \nu)\lambda' = \beta_i - \beta_j$. The finite nonzero denominator condition yields (36), and its numerator vanishes exactly when $\tau_i = \tau_j$. This is a local calculation on the specified branch. $\square$

D.2 Constant aggregate demand and strict convexity

Proof of Proposition 3. Let $\eta_D = \alpha_D + k_D$, $c_0 = \alpha_D(A - H_D)$ and $B = \sum_i \beta_i(H_i - q_i + d_i)$. On a regular branch the counterparty demand is $(\lambda - c_0)/\eta_D$. Clearing gives

$$\lambda = \frac{c_0 + \eta_D B}{1 + \eta_D S}, \quad R = \frac{B - S c_0}{1 + \eta_D S}. \quad (80)$$

When both $\beta_i = \beta$, $B = \beta(H_1 + H_2 - q_1 - q_2 + Q)$ is constant. So are $A, c_0, \lambda, R, c_D, z_D$. By the full-domain assumption these formulas describe actual equilibria throughout the comparison domain. This proves separation in (37).

At the constant equilibrium price, differentiation with respect to own termination gives

$$z'_i = -\tau, \quad c'_i = -\beta G_i, \quad c''_i = -k_i \beta^2.$$

For positive z the normal shortfall derivatives are $F_{\sigma,c} = -\pi$ and $F_{\sigma,z} = s$. Since $z''_i = 0$, the chain rule gives the Hessian quadratic term plus $F_{\sigma,c} c''_i = (-\pi_i)(-k_i \beta^2)$. Thus

$$f''_i = \frac{\phi(c_i/(\sigma z_i))}{\sigma z_i} \left(c'_i - \frac{c_i}{z_i} z'_i \right)^2 + \pi_i k_i \beta^2 > 0. \quad (81)$$

The second term is strictly positive because $\pi_i > 0$, $k_i > 0$ and $\beta > 0$. Hence the sum is strictly convex on every convex comparison domain with distinct allocations. It is continuous on the compact domain, so it attains a unique minimum (also trivially unique if the domain is a singleton).

The derivative formula gives $-f'_i = I_i^0$ in (38). For box bounds and a fixed sum, stationarity of $\sum_i f_i(d_i) + \mu_Q(\sum_i d_i - Q)$, together with the lower and upper bound multipliers, yields (39). Conversely those conditions imply, for every feasible $\tilde{d}$, $\sum_i f'_i(d_i)(\tilde{d}_i - d_i) \geq 0$. Convexity proves optimality, and strict convexity gives uniqueness. Fixed coordinates contribute no feasible direction. On domains with additional restrictions, the corresponding normal-cone conditions must be included. $\square$

D.3 Complete common-feedback formula

Retain the local conditions of Lemma 2. Define π_D, s_D from the counterparty's c_D, z_D in the same way as (38). The cash-intercept derivatives are

$$c'_\ell = -\beta_\ell G_\ell \delta_\ell + (v_\ell G_\ell - r_\ell) \lambda', \quad (82)$$

$$c'_D = [R + (\lambda - k_D R) \nu] \lambda'. \quad (83)$$

Each account contributes $-\pi c' + sz'$ to the derivative. Collect all coefficients on λ' in the quantity

$$\begin{aligned}\mathfrak{M} &= \sum_{\ell=1}^2 [\pi_\ell(r_\ell - v_\ell G_\ell) - v_\ell s_\ell] \\ &\quad - \pi_D[R + (\lambda - k_D R)\nu] + s_D \nu, \quad \Lambda = \frac{\mathfrak{M}}{S + \nu}.\end{aligned}\tag{84}$$

Substitution of (36) gives

$$\begin{aligned}D_{ij} &= (\beta_i \pi_i G_i - \tau_i s_i) - (\beta_j \pi_j G_j - \tau_j s_j) + (\beta_i - \beta_j) \Lambda \\ &= I_j - I_i, \quad I_i = \tau_i(s_i + \Lambda) - \beta_i \pi_i G_i.\end{aligned}\tag{85}$$

This includes counterparty risk and all induced cash transfers. $\mathfrak{M}$ is a feedback aggregate, distinct from cash M_j ; Λ is distinct from the allocation multiplier μ_Q . Computing it can require all participants' states. The formula by itself does not identify missing cash or compress the information needed before intervention.

On the regular branch $\nu = 1/(\alpha_D + k_D)$. On a smooth branch where the counterparty's quantity upper bound is the strictly active optimal response, $\nu = 0$. When its cash-margin upper bound is instead strictly active, differentiating that budget equality gives $\nu = -A/(m_D + k_D R)$, provided the denominator is nonzero and the bound is indeed its optimal response. These substitutions apply only while the hedgers remain slack and the response branch remains valid. If a hedger's cash upper bound is its locally optimal response and $m_i + k_i r_i \neq 0$, implicit differentiation of its budget at a fixed price gives $(m_i - \lambda)/(m_i + k_i r_i)$ as the rebuilding derivative. The original retention formula then no longer applies. No derivative at a zero denominator, or convexity or optimum claim for those constrained branches, is made here.

E Proof of Nonnormal Persistence

For the symmetric mixture (40), the CARA risk penalty for exposure y is

$$\rho_{j,\varepsilon}(y) = \frac{1}{\gamma_j} \log \left[(1 - \varepsilon) e^{\gamma_j^2 y^2 / 2} + \frac{\varepsilon}{2} e^{\gamma_j^2 y^2 / 4} + \frac{\varepsilon}{2} e^{3\gamma_j^2 y^2 / 4} \right]. \tag{86}$$

This uses the log moment-generating function, not a mean-variance approximation. Symmetry makes the penalty the same for exposure y and $-y$. The private objectives become

$$\begin{aligned}U_{i,\varepsilon} &= E_i - \lambda r_i - \frac{1}{2} k_i r_i^2 - \rho_{i,\varepsilon}(H_i - a_i - r_i), \\ U_{D,\varepsilon} &= b + \lambda R - \frac{1}{2} k_D R^2 - \rho_{D,\varepsilon}(A + R - H_D).\end{aligned}\tag{87}$$

All budgets and terminal-account identities remain those of Section 2.

The second derivative of a log moment-generating function is a tilted variance. Hence $\rho''_{j,\varepsilon} \geq 0$ for the probability mixture and $k_j + \rho''_{j,\varepsilon} > 0$. The costs make each private objective strictly concave and coercive in its own unconstrained trade, giving a unique unconstrained maximizer. The regular first-order conditions are

$$\begin{aligned}-\lambda - k_i r_i + \rho'_{i,\varepsilon}(H_i - a_i - r_i) &= 0, \\ \lambda - k_D R - \rho'_{D,\varepsilon}(A + R - H_D) &= 0, \quad r_1 + r_2 = R.\end{aligned}\tag{88}$$

Cash b is absent from these equations.

Proof of Proposition 4. We first continue the equilibrium uniformly in u . On a bounded neighborhood of the original solution curve, the logarithm's argument in (86) is positive near $\varepsilon = 0$ and all required derivatives are smooth. A local mathematical extension to slightly negative ε can be used for the implicit function theorem; only nonnegative probability mixtures are asserted as economies.

At fixed u, ε , differentiating the private FOCs with respect to price gives

$$\partial_\lambda r_i = -\frac{1}{k_i + \rho''_{i,\varepsilon}}, \quad \partial_\lambda R = \frac{1}{k_D + \rho''_{D,\varepsilon}}. \quad (89)$$

Thus the clearing excess $r_1 + r_2 - R$ has strictly negative price derivative. At $\varepsilon = 0$, it equals $-1/4 - 1/4 - 50 = -101/2$ everywhere on the original compact solution curve. The parameter implicit function theorem therefore gives a smooth continuation near each u . A finite cover of $[0, 1]$ provides a common positive neighborhood of $\varepsilon = 0$. On the nonnegative side, unique private optima and strictly decreasing excess demand make overlapping local solutions agree, yielding a uniform smooth branch on the entire interval. Choose the continuation bound at most one so that all asserted laws remain probability mixtures.

Theorem 1(1) supplies strictly positive uniform slack for cash, quantity and final-position constraints at the original cash pair. Equilibrium variables depend continuously on ε , uniformly on the compact allocation interval. Decreasing the common neighborhood if necessary preserves all of these slacks. The unconstrained optima then remain actual constrained optima. Because b does not enter (88), the same feasible solution works for both cash states; its maps and all existing derivatives coincide across them.

For fixed c and $z > 0$, the expected shortfall under the mixture is exactly

$$F_\varepsilon(c, z) = (1 - \varepsilon)F_1(c, z) + \frac{\varepsilon}{2}F_{\sqrt{1/2}}(c, z) + \frac{\varepsilon}{2}F_{\sqrt{3/2}}(c, z). \quad (90)$$

The law is symmetric, so this formula applies to both sides of the contracts. Positions on the continued branch stay uniformly away from zero, and cash intercepts and their needed derivatives are bounded for the two fixed cash states. Substituting the smooth equilibrium into (90) gives uniform continuity of $J'_{\varepsilon,b}$ and $J''_{\varepsilon,b}$ in ε over $u \in [0, 1]$.

The original curvature is uniformly greater than $21/400$, and the two point derivatives have the strict margins in (22)–(23). Reduce ε_0 once more so that the curvature changes by less than $21/800$ and both signs at $u = 1/4$ persist. Then $J''_{\varepsilon,b} > 21/800$ on the full interval. Continuity, compactness and strict convexity give unique minimizers; the opposite derivative signs place them strictly on opposite sides of $1/4$, including the possibility of endpoint minima. Equality of the permitted observations makes the allocation-rule conclusion immediate, as in Appendix A.5. The opposite signs also require opposite strict local directions at the comparison point. $\square$

The distributional moments follow directly from the component normal moments. The variance is $(1 - \varepsilon) + \varepsilon/4 + 3\varepsilon/4 = 1$, and the fourth moment is $3(1 - \varepsilon) + 3\varepsilon/8 + 27\varepsilon/8 = 3 + 3\varepsilon/4$. This verifies that the class is nonnormal for every positive ε . No quantitative neighborhood size or numerical nonnormal equilibrium has been asserted.

F Correspondence of the Private Trading Problem

This appendix makes the literature comparison at a fixed allocation explicit. It retains positive quadratic execution costs and concerns the unconstrained private problem, or its regular strictly feasible solution. It does not identify the complete regulatory objective with a Radner market-cost or XVA criterion.

Let $\eta_j = \alpha_j + k_j$ and define

$$h_i = \frac{\alpha_i H_i + k_i a_i}{\eta_i}, \quad h_D = \frac{\alpha_D H_D + k_D A}{\eta_D}. \quad (91)$$

Substitute $r_i = z_i - a_i$ into (11). Completing the square gives

$$\begin{aligned} U_i &= \mathcal{C}_i - \lambda z_i - \frac{\eta_i}{2}(z_i - h_i)^2, \\ \mathcal{C}_i &= E_i + \lambda a_i - \frac{k_i a_i^2 + \alpha_i H_i^2}{2} + \frac{\eta_i h_i^2}{2}. \end{aligned} \quad (92)$$

Similarly, substituting $R = z_D - A$ into (12),

$$\begin{aligned} U_D &= \mathcal{C}_D + \lambda z_D - \frac{\eta_D}{2}(z_D - h_D)^2, \\ \mathcal{C}_D &= b - \lambda A - \frac{k_D A^2 + \alpha_D H_D^2}{2} + \frac{\eta_D h_D^2}{2}. \end{aligned} \quad (93)$$

At given allocation and price, $\mathcal{C}_j$ does not depend on that participant's choice. Hence

$$z_i = h_i - \frac{\lambda}{\eta_i}, \quad z_D = h_D + \frac{\lambda}{\eta_D}, \quad \lambda = \frac{h_1 + h_2 - h_D}{\eta_1^{-1} + \eta_2^{-1} + \eta_D^{-1}}. \quad (94)$$

In the entropic Gaussian model of Bastide et al. (2024, Proposition 2.4, equation (2.12)), take traded payoff X , price $-\lambda$, signed positions $(-z_1, -z_2, z_D)$, risk endowments $(h_1 X, h_2 X, -h_D X)$ and entropic coefficients η_j/σ^2 . The private certainty equivalents then produce (94). The fixed-total intervention generates a family of these trading problems as d changes.

The constants in (92)–(93) can depend on allocation. The actual funding intercepts remain $E_i - \lambda r_i - k_i r_i^2/2$ and $b + \lambda R - k_D R^2/2$; they cannot be replaced by the square-completion constants. Strict feasibility must still be verified under the original budgets. These distinctions explain why the private-demand correspondence alone neither proves nor refutes equivalence of complete allocation objectives.

Cash translation itself is also shared with the entropic framework: adding deterministic cash shifts the certainty equivalent by a constant and leaves unconstrained private choice unchanged. The whole-domain feasibility checks and the disjoint-minimizer argument are separate steps in the present construction.

G Notation

Symbol	Meaning
$i = 1, 2; D$	The two hedgers; the surviving risk-bearing counterparty.
$Q, d, \mathcal{D}$	Fixed total termination; its allocation; the fixed-sum allocation domain.
u	Allocation coordinate in the construction: $d = (u, 1 - u)$, $u \in [0, 1]$.
q_i, a_i, A	Initial shorts; remaining old shorts $q_i - d_i$; their fixed total, also the counterparty's remaining old longs.
H_i, H_D	Outside exposure coefficients: $H_i X$ and $-H_D X$.
P_0, P_1, P_2	Reference, date-1 and terminal price levels; only their differences enter payoffs.
X, σ	Terminal payoff change $P_2 - P_0$ and its normal standard deviation.
λ	$P_0 - P_1$, the equilibrium date-1 price adjustment.
E_i, b	Initial counted hedger cash; initial counterparty cash. The latter is the only differing primitive in the main construction.
r_i, R, K	Additional short trades; total additional longs; the net-additional-long cap.
z_i, z_D	Final short and long positions.
k_j, m_j	Quadratic execution-cost coefficient and cash margin per contract.
M_j	Counted cash after date-1 marking and execution costs, before final net settlement; includes committed and free cash.
c_j	Terminal account resource intercept; not date-1 available cash.
e_j	Terminal account balance under contractual settlement, before outside replenishment.
o_j, ℓ_j	Signed final net payment obligation; nonnegative outside top-up in the accounting identity.
γ_j, α_j	CARA risk-aversion coefficient; $\alpha_j = \gamma_j \sigma^2$.
η_j, v_i	$\eta_j = \alpha_j + k_j$; $v_i = 1/\eta_i$, the magnitude of the hedger's unconstrained price response.
β_i, τ_i	Fixed-price rebuilding response α_i/η_i ; retention share $k_i/\eta_i = 1 - \beta_i$.
$U_j, \mathcal{F}_j$	Private certainty equivalent and feasible trading set.
$J(d; b), J_b(u)$	Sum of expected account shortfalls on the specified branch; its one-dimensional form in the construction.
F_σ, ϕ, Φ	Normal account-shortfall function; standard normal density and CDF.
π_j	Account funding-shortfall probability, not unrecoverable-default probability.
s, θ_b	Known fixed structure excluding counterparty cash; full state (s, b) .
$x_{\text{reg}}, \mathcal{O}, \psi$	Specified regular price/trade/position map; permitted observation; observation-based allocation rule.
$\delta, D_\delta J$	A feasible fixed-total direction; the associated directional derivative.
u_b^*	Unique minimizer on the designated regular branch, not across all equilibrium selections.
T, P_T	$T = \text{span}(\mathcal{D} - d_0)$ at a relative interior allocation; its orthogonal projection.

Symbol	Meaning
J_H, V, Y_δ	Hedgers' aggregate expected shortfall; counterparty resources excluding initial cash; $Y_\delta = D_\delta V$.
$a_\delta, g_\delta, C_\delta$	Hedger contribution, total objective derivative and counterparty contribution in direction δ .
$m_\delta(t), f_V(t)$	Conditional mean $\mathbb{E}[Y_\delta \mid V = t]$ and density of the resource V . The former is not margin m_j .
$\kappa, z, t(b)$	Intercept excluding cash, long exposure and shortfall cutoff for $V = \kappa + zX$; here $z = z_D$.
F_X, L_X	Payoff CDF and tail first moment $L_X(t) = \mathbb{E}[X \mathbf{1}\{X < t\}]$.
$b^*(d_0, \delta; s)$	Cash threshold for a specified local priority under the stated tail and crossing conditions.
S, ν	Sum $v_1 + v_2$; the counterparty's price-response slope on the selected branch.
s_j, G_i, I_i^0	Marginal shortfall cost of position; $\lambda + k_i r_i$; own marginal allocation benefit at unchanged price.
$\mu_Q, \mathfrak{M}, \Lambda$	Outer fixed-total multiplier; aggregate local price-effect coefficient; common feedback $\mathfrak{M}/(S + \nu)$.
$\varepsilon, \rho_{j,\varepsilon}$	Mixture perturbation size and its CARA risk penalty.
$h_j, \mathcal{C}_j$ in Appendix F	Effective Gaussian risk endowment coefficient and square-completion constant, not terminal funding intercept c_j .

H Reproduction and Figure Inputs

The replication package is available from the author at zc2988@nyu.edu. File paths below refer to the unpacked package.

All figures use `original_parameters.json`, the two-case input file in the original-witness directory listed below. Table 1 gives these same cases. The only differing primitive is counterparty cash. There is no empirical calibration, parameter search or numerical nonnormal exercise.

File or directory	Purpose
<code>replication/ original_witness/</code>	Original rational certificate, parameters and recorded baseline run.
<code>replication/ verify_paper_certificate.py</code>	Exact check of the paper’s rational input table, curvature arithmetic and normal-integral enclosures.
<code>replication/ verify_algebra_identities.py</code>	Formal-polynomial checks of the cash-flow, private-choice, local-feedback and square-completion identities.
<code>replication/make_figures.py</code>	Recreates both figures and the local decomposition table from the original parameter file.
<code>figure_data/ allocation_curves.csv</code>	Price, trade, position, objective and derivative values on the display grid.
<code>figure_data/ local_decomposition.csv</code>	Hedger and counterparty contributions at $u = 1/4$.
<code>verification/ figure_results.json</code>	Display settings, numerical minimum method, software versions and consistency checks.

The figure grid has 401 allocations on $[0, 1]$. It renders explicit functions rather than proving feasibility or convexity by sampling. Normal probabilities are evaluated using the complementary error function. For the displayed minimum, the code first checks the signs of $J'_b(0)$ and $J'_b(1)$; only if those signs bracket an interior root does it bisection the strictly increasing derivative. Those numerical locations are display outputs, not additional analytical theorems.

To recreate the figures from the supplied directory, run

```
python3 replication/make_figures.py
```

with the versions of NumPy, Matplotlib and its dependencies recorded in the package’s requirements file. The figure outputs are supplied in PDF, SVG and PNG, with CSV inputs and a hash manifest. The exact arithmetic verifier uses the Python standard library. Build the paper with `python3 verification/build_paper.py` after installing Tectonic; the editable inputs are `main.tex`, `sections/`, `appendices/` and `references.bib`.

The algebra checker uses formal variables and exact rational coefficients, without choosing economic parameters. Its identities complement the proofs; it does not establish their regularity or equilibrium-existence conditions. The nonnormal result is supported by the analytic continuation proof in Appendix E, not by a numerical choice of perturbation size. The common-response and rational-sign arguments retain their original parameters throughout.